Can Ideas Behind Ancient Egyptian Fractions Speed up Modern Computers?

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Abstract

To divide two numbers a and b, modern computers use an algorithm which is more efficient that what we humans normally do: they compute $a \cdot (1/b)$, where for all sufficiently small integers b, the inverse $1/b$ is pre-computed. For fractions, when both a and b are integers, this algorithm requires only one multiplication. Can we make the procedure even faster by not using multiplication at all? To do this, we need to represent each fraction as the sum of inverses – which, interestingly, is how ancient Egyptians represented fractions.

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1 Formulation of the Problem

In computers, all computations are reduced to arithmetic operations. Computers perform many important computations. We can use them to compute the value of complex functions, we can use them to solve complex partial differential equations, etc. However, the only operations which are supported on the hardware level are arithmetic operations. All other computations – no matter how complex they are – are performed, in a computer, as a sequence of arithmetic operations.

In some cases, a translation of a computational task into a sequence of arithmetic operations is straightforward. For example, to compute $\exp(x)$ or $\sin(x)$, we expand these functions in Taylor series and keep several first terms in this expansion. As a result, the desired function is approximated by a polynomial, and a polynomial is, by definition, what can be obtained by variables and constants by using addition, subtraction, and multiplications.

How arithmetic operations are implemented in a computer. In general, there are four basic arithmetic operations – addition, subtraction, multiplication, and division. For us – and for a computer – the simplest are addition and subtraction. Multiplication is more complex because, e.g., for binary numbers, the usual multiplication algorithm – in which we multiply one of the numbers by digits and then add the result – reduced to several additions: indeed, binary digits are 0 and 1, so multiplying by these digits is trivial.

Division is even more complex than multiplication, since the usual way – that we learn at school – requires several multiplications. In a computer, division a/b is implemented in a more efficient way: as $a/b = a \cdot (1/b)$, where the inverses $1/b$ are:

- pre-recorded for small integers b, and
- computed based on these pre-recorded values for non-integer denominators b.

This way, computing each rational number – i.e., each fraction a/b in which both a and b are natural numbers – requires only one multiplication.

Natural question: can we do better? Can we make this computation even faster? We have already went from the traditional division algorithm – that requires several multiplications – to an algorithm that needs only one multiplication. So, the only way to further speed up computations is to find a way to avoid multiplication altogether.

What we do in this paper. In this paper, we show that this may indeed be possible – if we use an idea going back several thousand years, to ancient Egyptian fractions.

2 Egyptian Fractions and the Resulting Idea

What we want. We want to avoid multiplication. We still need to use $1/b$ values – otherwise, if we only use addition and subtraction, we will never get beyond integers. We do not want to apply multiplication to these fractions, we only want to apply addition (and subtraction, but subtracting $1/b$ is equivalent to adding $1/(-b)$). In other words, we want to represent the fraction a/b as the sum of inverses:

$$\frac{a}{b} = \frac{1}{n_1} + \ldots + \frac{1}{n_k}. \tag{1}$$

This is exactly what ancient Egyptians did. Interestingly, this is exactly how people in ancient Egypt represented fractions: as sums of inverses; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

How exactly we can derive such a representation: idea. Of course, we can always represent the fraction a/b as

$$\frac{a}{b} = \frac{1}{b} + \ldots + \frac{1}{b} \text{ (}a\text{ times)}, \tag{2}$$

but for large a, this would require too many additions – and the resulting computation time will be large. We therefore need to find a representation (1) with a small number of terms.

A natural idea is thus – after we sort the inverses in order, so that

$$n_1 \le n_2 \le \ldots \le n_k$$

and thus, $1/n_1 \ge \ldots \ge 1/n_k$ – is to select the first term $1/n_1$ to be as large as possible – i.e., to select the value n_1 ro be as large as possible.

The condition $a/b \ge 1/n_1$ is equivalent to $n_1 \ge b/a$. Thus, the smallest possible n_1 with this property is the smallest possible integer which is larger than or equal to b/a. Such an integer is known as a *ceiling* of the value b/a; it is denoted by $n_1 = \lceil b/a \rceil$.

If a divides b, i.e., if $b = c \cdot a$ for some c, then $a/b = 1/c$, i.e., we get the representation (1) with just one value $n_1 = c$. In general, a does not divide b, i.e., we have $b = c \cdot a + r$ for some non-zero remainder $0 < r < a$. In this case, $b/a = c + r/a$, where $0 < r/a < 1$, and so $n_1 = \lceil b/a \rceil = c + 1$.

Now, the remaining part $a/b - 1/c$ has the form

$$\frac{a}{b} - \frac{1}{c} = \frac{a}{c \cdot a + r} - \frac{1}{c+1} = \frac{(c+1) \cdot a - (c \cdot a + r)}{b \cdot (c+1)} = \frac{a-r}{b \cdot (c+1)}.$$

The numerator $a - r$ of this remaining part is smaller than the original numerator a.

The remainder r can take any value between 0 and a, so its average value is $a/2$. Thus, on average, the numerator of the remaining part is twice smaller than the original numerator a. In 2 steps, we get a numerator $a/4$, in k steps, we get a numerator of order $a/2^k$. So, in $k = \log_2(a)$ steps, when $2^k = a$, we get the numerator equal to $a/2^k = 1$ – i.e., the remaining fraction is the inverse of an integer. Thus, on average, we only need $\log_2(a)$ terms in the expression (1). This number is much smaller than a terms needed for the naive representation (2).

It is not clear whether the new algorithm is always better. Whether the use of Egyptian fraction will indeed be faster than multiplication is not clear. Indeed:

- The use of Egyptian fractions requires $\log_2(a)$ additions.
- On the other hand, multiplication of a binary number by a means adding as many terms as there are binary digits in a – i.e., also, an order or $\log_2(a)$.

So, to decide which method is faster we need to implement the new method and check the results.

How to derive the desired representation: algorithm. If a divides b, i.e., if $b = c \cdot a$ for some c, then $a/b = 1/c$ is the desired 1-term representation of type (1).

If a does not divide b, then we take $n_1 = \lceil b/a \rceil$. To the remaining fraction $a/b - 1/n_1$, we apply the same algorithm, etc., until we get a representation of type (1).

Examples. Let us consider, as examples, all irreducible fractions a/b with $b \le 7$ and $1 < a < b$.

For $b = 3$, the only such fraction is $2/3$. Here, $\lceil 3/2 \rceil = 2$, so $n_1 = 2$. The difference $2/3 - 1/2$ is equal to $1/6$, so we get

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}.$$

For $b = 4$, the only such fraction is $3/4$. By applying the above algorithm, we get

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}.$$

For $b = 5$, we have three such fractions: $2/5$, $3/5$. and $4/5$. Here,

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{15},$$

and

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}.$$

For $4/5$, we get $n_1 = 2$. Here, $4/5 - 1/2 = 3/10$. For $3/10$, we get $n_2 = \lceil 10/3 \rceil = 4$, and $3/10 - 1/4 = 1/20$, so

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}.$$

For $b = 6$, the only such fraction is 5/6:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}.$$

For $b = 7$, we get

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28},$$

$$\frac{3}{7} = \frac{1}{3} + \frac{2}{21} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231};$$

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14},$$

$$\frac{5}{7} = \frac{1}{2} + \frac{3}{14} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70},$$

$$\frac{6}{7} = \frac{1}{2} + \frac{5}{14} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

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