Halting Problem

Discussion. Up to now, we analyzed what can be computed and on what computational device. However, there are some things that cannot be computed. One of these things is checking whether a program will halt on given data.

Of course, if a program halts, we know it halted, but what if it does on and on? Maybe it will eventually halt – or maybe it will never halt? It turns out that in the general case, it is not possible to tell whether the program will eventually halt or not.

We do not teach this result to incoming students, since we want to preserve the right to lower their grades if their programs do not halt – just kidding. Seriously, for simple programs, it is possible to check whether a program halts, but, as we will show, not in the general case.

Let us now show the proof.

Theorem. No algorithm is possible that, given a program p and data d, checks whether p halts on d.

Proof. We will prove this result by contradiction. Let us assume that there exists an algorithm (i.e., a Java program) “halt-checker”s that:

- given two strings: a program p and data d,
- returns true if p halts on d, and false otherwise.

Let us now build the following auxiliary program:

```java
public state int aux(String x){
    if (halt-checker(x,x)) {
        while(True) x = x;
    } else {return 0;}
}
```

Will this program aux halt if, as input, we give it the same string aux?

- If aux halts on aux, then, by definition of the halt-checker, the value
  
  halt-checker(aux,aux)

  is true. If you trace the above program aux, you will see that in this case, this program will go into an infinite loop – and thus, it will not halt.
• On the other hand, if aux does not halt on aux, then, by definition of the halt-checker, halt-checker(aux,aux) is false. If you trace the above program aux, you will see that in this case, this program will halt – namely, it will return 0.

In both case, we get a contradiction:

• if we assume that it halts, we conclude that it does not halt, and

• if we assume that it does not halt, then we conclude that it halts.

This contradiction shows that halt-checkers are indeed not possible. The theorem is proven.

Comment. Not only we have a theoretical proof, but also:

• if someone presents us with a program that supposedly solves the halting program,

• we can immediately produce an example on which this presented program will not work correctly – namely, the above-constructed program aux.