Problem 1. Finite automata and regular languages.

Problem 1a. Design a finite automaton for recognizing non-empty words that start with the letter $a$. Assume that the input strings contain only symbols $a$ and $p$. The easiest is to have three states:

- the starting state $s$ corresponding to the empty input string,
- the state $f$ indicating that the first symbol that we have read was $a$, and
- the state $e$ indicating all other words.

You just need to describe transitions between these states, and which states are final. Show, step-by-step, how your automaton will accept the word $app$.

Solution.

- from $s$, $a$ leads to $f$, and $p$ leads to $e$;
- from $f$, both symbols lead to $f$;
- from $e$, both symbols lead to $e$.

The final state is $f$. Derivation of $app$ is as follows:

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<tr>
<th></th>
<th>$a$</th>
<th>$p$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$f$</td>
<td>$f$</td>
<td>$f$</td>
</tr>
</tbody>
</table>
Problem 1b. Explain why in most computers binary numbers are represented starting with the lowest possible digit.

Solution. In most actual computers, the representation of a number starts with the least significant digit, since all arithmetic operations like addition, subtraction, or multiplication start with the least significant digit. So, if we store the number the way we write numbers, most significant digits first, computers will have to waste time going through all the digits until they come up with the least significant digit and start the actual computations. To speed up computations, representations therefore start with the least significant digits.
Problem 1c. On the example of the above automaton, show how the word *app* can be represented as $xyz$ in accordance with the pumping lemma.

Here, the first repeating state is $f$:

\[
\begin{array}{c|c|c|c|c}
| & a & p & p & f \\
\hline
s & f & f & f & f \\
\end{array}
\]

So, $x = a$, $y = p$, and $z = p$.  

**Problem 1d.** Use a general algorithm to describe a regular expression corresponding to the finite automaton from the Problem 1a. (If you are running out of time, it is Ok not to finish, just eliminate the first state.)

**Solution.** We start with the described automaton:

```
According to the general algorithm, first we add a new start state ns and a few final state f, and we add jumps:
- from the new start state ns to the old start state, and
- from each old final state to the new final state nf.
```

As a result, we get the following automaton.

```
Then, we need to eliminate the three intermediate states s, f, and e one by one. Let us start with eliminating the state e. In genera, we have a formula
\[
R'_{i,j} = R_{i,j} \cup (R_{i,k}R_{k,k}^*R_{k,j}),
\]
where k is the state that we are eliminating. In this case, we have k = e, so
\[
R'_{i,j} = R_{i,j} \cup (R_{i,e}R_{e,e}^*R_{e,j}).
\]
From the state e, we cannot go anywhere, so \(R_{e,j} = \emptyset\) for all \(j\). Thus,
\[
R'_{i,j} = R_{i,j} \cup (R_{i,e}R_{e,e}^*\emptyset) = R_{i,j} \cup \emptyset = R_{i,j}.
\]
Thus, we can simply delete the state e without changing anything else:
Let us then eliminate the state $f$. Then, we have the following:

\[
R'_{ns,s} = R_{ns,s} \cup (R_{ns,f}R_{f,f}R_{f,s}) = \Lambda \cup (\emptyset \ldots) = \Lambda \cup \emptyset = \Lambda;
\]
\[
R'_{ns,nf} = R_{ns,nf} \cup (R_{ns,f}R_{f,f}R_{f,nf}) = \emptyset \cup (\emptyset \ldots) = \emptyset \cup \emptyset = \emptyset;
\]
\[
R'_{s,s} = R_{s,s} \cup (R_{s,f}R_{f,f}R_{f,s}) = \emptyset \cup (a(a \cup p)^*\emptyset) = \emptyset \cup \emptyset = \emptyset;
\]
\[
R'_{s,nf} = R_{s,nf} \cup (R_{s,f}R_{f,f}R_{f,nf}) = \emptyset \cup (a(a \cup p)^*\Lambda) = \emptyset \cup (a(a \cup p)^*) = a(a \cup p)^*.
\]
Thus, the 3-state $a$-automaton takes the following form:

![Diagram of the 3-state automaton](image)

The final expression is the corresponding expression for $R'_{ns,nf}$:

\[
R'_{ns,nf} = R_{ns,nf} \cup (R_{ns,s}R_{s,s}R_{s,nf}) = \\
\emptyset \cup (\Lambda \emptyset^* a(a \cup p)^*) = \emptyset \cup (\Lambda \Lambda a(a \cup p)^*) = \\
\emptyset \cup (a(a \cup p)^*) = \\
a(a \cup p)^*.
\]

The formula on the previous line is a regular expression corresponding to the original automaton.
**Problem 1e-f.** The resulting language can be described by a regular expression $a(a \cup p)^*$. Use a general algorithm to transform this regular expression into a finite automaton: first a non-deterministic one, then a deterministic one.

**Solution.** We start with the standard non-deterministic automata for recognizing the words $a$ and $p$:

Then, we use the general algorithm for the union to design a non-deterministic automaton for recognizing the language $a \cup p$:

Now, we apply a standard algorithm for the Kleene star, and we get the following non-deterministic automaton for $(a \cup p)^*$:

Now, we also take a standard automaton for the language $a$, and use the algorithm for concatenation for combine them:
To get a deterministic finite automation, first, we enumerate the states:

Then, we get the following deterministic automaton:
Problem 2. *Beyond finite automata: pushdown automata and context-free grammars*

Problem 2a. A fictitious town of Osaple consists of three districts A, B, and C. Districts A and B have the same number of people, while District C has twice as many folks. When representative from these districts meet to make a city-wide decision, they want to make sure that A and B has the same number of representative, while C has twice as many. So, e.g., if we have representative CABCABCC, this is fair. Prove that the set of all such “fair” sequences is not regular and therefore, cannot be recognized by a finite automaton.

Solution We will prove it by contradiction. Let us assume that the language $L$ of all fair sequences is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer $p$ such that every word from $L$ whose length $\text{len}(w)$ is at least $p$ can be represented as a concatenation $w = xyz$, where:

- $y$ is non-empty;
- the length $\text{len}(xy)$ does not exceed $p$, and
- for every natural number $i$, the word $xy^i z \overset{\text{def}}{=} xy \ldots yz$, in which $y$ is repeated $i$ times, also belongs to the language $L$.

Let us take the word

$$w = A^p B^p C^{2p} = A \ldots AB \ldots BC \cdot C,$$

in which first $A$ is repeated $p$ times, then $B$ is repeated $p$ times, and finally $C$ is repeated $2p$ times. The length of this word is $p + p + 2p = 4p > p$. So, by pumping lemma, this word can be represented as $w = xyz$ with $\text{len}(xy) \leq p$. This word starts with $xy$, and the length of $xy$ is smaller than or equal to $p$. Thus, $xy$ is among the first $p$ symbols of the word $w$ – and these symbols are all $A$’s. So, the word $y$ only has $A$’s.

Thus, when we go from the word $w = xyz$ to the word $xyyz$, we add $A$’s, and we do not add any $B$’s. So, in the word $xyyz$, there are more $B$’s than $A$’s. Thus, the word $xyyz$ cannot be in the language $L$, since by definition $L$ only contains words which have exactly as many $B$’s as $A$’s.

On the other hand, by Pumping Lemma, the word $xyyz$ must be in the language $L$. So, we proved two opposite statements:

- that this word is *not* in $L$ and
- that this word is in $L$.

This is a contradiction.

The only assumption that led to this contradiction is that $L$ is a regular language. Thus, this assumption is false, so $L$ is not regular.
**Problem 2b.** Use a general algorithm to transform the finite automaton from the Problem 1a into a context-free grammar (CFG). Show, step-by-step, how this CFG will generate the word *app*.

**Solution.** According to the general algorithm, the corresponding grammar should have three variables $S$, $F$, and $E$, and the following rules:

$$S \rightarrow aF, \quad S \rightarrow pE, \quad F \rightarrow aF, \quad F \rightarrow pF, \quad E \rightarrow aE, \quad E \rightarrow pE, \quad F \rightarrow \varepsilon.$$  

The word *app* is accepted by the finite automaton as follows:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>p</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>f</td>
<td>f</td>
<td>f</td>
</tr>
</tbody>
</table>

Thus, its derivation takes the following form:

$$S \rightarrow aF \rightarrow apF \rightarrow appF \rightarrow app.$$
**Problem 2c.** For the context-free grammar from the Problem 2b, show how the word *app* can be represented as *uvxyz* in accordance with the pumping lemma.

**Solution.** In terms of a tree, the derivation of the word *app* can be represented as follows:

```
    S
   / \  
  a   F
    / \  
   p   F
      / \  
     p   F
        / \  
       ε  
```

Here, $u = ap$, $v = p$, $x = y = z = ε$. 
Problem 2d. Use a general algorithm to translate the CFG from 2b into Chomsky normal form.

Solution.

Preliminary step. We add a new starting variable $S_0$ and a rule $S_0 \rightarrow S$:

$$S \rightarrow aF, \quad S \rightarrow pE, \quad F \rightarrow aF, \quad F \rightarrow pF, \quad E \rightarrow aE, \quad E \rightarrow pE, \quad F \rightarrow \varepsilon, \quad S_0 \rightarrow S.$$ 

Step 0. We need to eliminate the rule $F \rightarrow \varepsilon$. This means adding the rules $S \rightarrow a$, $F \rightarrow a$, and $F \rightarrow p$:

$$S \rightarrow aF, \quad S \rightarrow pE, \quad F \rightarrow aF, \quad F \rightarrow pF, \quad E \rightarrow aE, \quad E \rightarrow pE, \quad S_0 \rightarrow S,$$

$$S \rightarrow a, \quad F \rightarrow a, \quad F \rightarrow p.$$

Step 1. We eliminate the rule $S_0 \rightarrow S$ by adding the rules $S_0 \rightarrow aF$, $S_0 \rightarrow pE$, and $S_0 \rightarrow a$:

$$S \rightarrow aF, \quad S \rightarrow pE, \quad F \rightarrow aF, \quad F \rightarrow pF, \quad E \rightarrow aE, \quad E \rightarrow pE,$$

$$S \rightarrow a, \quad F \rightarrow a, \quad F \rightarrow p, \quad S_0 \rightarrow aF, \quad S_0 \rightarrow pE, \quad S_0 \rightarrow a.$$

Step 2. We introduce two new variables $V_a$ and $V_p$, replace $a$ and $p$ in length-2 right-hand sides with $V_a$ and $V_p$, and add rules $V_a \rightarrow a$ and $V_p \rightarrow p$:

$$S \rightarrow V_a F, \quad S \rightarrow V_p E, \quad F \rightarrow V_a F, \quad F \rightarrow V_p F, \quad E \rightarrow V_a E, \quad E \rightarrow V_p E,$$

$$S \rightarrow a, \quad F \rightarrow a, \quad F \rightarrow p, \quad S_0 \rightarrow aF, \quad S_0 \rightarrow pE, \quad S_0 \rightarrow a, \quad V_a \rightarrow a, \quad V_p \rightarrow p.$$ 

This is already Chomsky normal form.
Problem 2e. Use a general algorithm to translate the CFG from 2b into an appropriate push-down automaton. Explain, step-by-step, how this automaton will accept the word app.

Solution.

The word app is derived as follows:

\[
S \rightarrow aE \rightarrow apE \rightarrow appE \rightarrow app;
\]

So, we have the following acceptance by the pushdown automaton:

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<th>a_1</th>
<th>w</th>
<th>a_4</th>
<th>w</th>
<th>a_4</th>
<th>w</th>
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<th>a_4</th>
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Problem 2f. Use the general stack-based algorithms to show:

- how the compiler will transform a Java expression $14/7/3$ into inverse Polish (postfix) notation, and
- how it will compute the value of this expression.

Solution. Let us show, step by step, how the above expression is transformed into the postfix form:

$$
\begin{array}{c}
14 & / & 7 & / & 3 \\
14 & 7 & / & 3 & / \\
/ & / & / & /
\end{array}
$$

Let us now show how this expression will be computed:

$$
\begin{array}{c}
14 & 7 & / & 3 & / \\
14 & 7 & 2 & 3 & 0 \\
14 & 2
\end{array}
$$
Problem 3. Beyond pushdown automata: Turing machines

Problem 3a. Prove that the language of fair sequences as described in Problem 2a is not context-free and therefore, cannot be recognized by a pushdown automaton.

Solution: Proof by contradiction. Let us assume that this language is context-free. Then, by the pumping lemma for context-free grammars, there exists an integer \( p \) such that every word \( w \) from this language whose length is at least \( p \) can be represented as \( w = uvxyz \), where \( \text{len}(vy) > 0 \), \( \text{len}(vxy) \leq p \), and for every \( i \), we have \( uv^i xy^i z \in L \).

Let us take the word \( w = A^p B^p C^{2p} \in L \), in which first we have \( A \) repeated \( p \) times, then \( B \) repeated \( p \) times, and then \( C \) repeated \( 2p \) times. The length of this word is \( p + p + 2p = 4p > p \), so this word can be represented as \( w = uvxyz \).

Where is \( vxy \)? Since the length of this part does not exceed \( p \), this word cannot contain \( A \), \( B \), and \( C \) – otherwise, it will have to contain all the symbols \( B \) between \( A \)s and \( C \)s – there are \( p \) of these symbols – and also at least one \( A \) and at least one \( C \). So, we have the following possible cases:

- \( vxy \) is in \( A \)
- \( vxy \) is between \( A \)s and \( B \)s
- \( vxy \) is in \( B \)
- \( vxy \) is between \( B \)s and \( C \)s
- \( vxy \) is in \( C \)

In the first case, \( v \) and \( y \) contain only \( A \). So, when we go from \( uvxyz \) to \( uvvxyyz \), we add \( A \)s, but we do not add \( B \)s and \( C \)s; thus, the desired balance between numbers of \( A \)s, \( B \)s, and \( C \)s is disrupted, and so \( uvvxyyz \notin L \) – while by pumping lemma, we should have \( uvvxyyz \in L \). Thus, this case is impossible.

In the second case, \( v \) and \( y \) contain only \( A \)s and \( B \)s. So, when we go from \( uvxyz \) to \( uvvxyyz \), we add \( A \)s and \( B \)s, but we do not add any \( C \)s; thus, the desired balance between numbers of \( A \)s, \( B \)s, and \( C \)s is disrupted, and so \( uvvxyyz \notin L \) – while by pumping lemma, we should have \( uvvxyyz \in L \). Thus, this case is impossible.

Similarly, we can prove that the other three cases are also not possible. So, none of the cases is possible, which means that our assumption that the language \( L \) is regular is wrong.
Problem 3b-c. Use a general algorithm to design a Turing machine that accepts exactly all sequences accepted by a finite automaton from Problem 1a. Show, step-by-step, how this Turing machine will accept the word $app$. Describe, for each step, how the state of the tape can be represented in terms of states of two stacks.

Solution: This Turing machine will have the following rules:

- start, $\rightarrow s$, R
- $s$, $a \rightarrow f$, R
- $s$, $p \rightarrow e$, R
- $f$, $a \rightarrow f$, R
- $f$, $p \rightarrow f$, R
- $e$, $a \rightarrow e$, R
- $e$, $p \rightarrow e$, R
- $s$, $\rightarrow$ reject
- $f$, $\rightarrow$ accept
- $e$, $\rightarrow$ reject

Tracing:

Moment 1:

| - | a | p | p | $\ldots$ | start |

Here, the left stack is empty, the right stack has the following form:

- $a$
- $p$
- $p$

Moment 2:

| - | a | p | p | $\ldots$ | s |

Here, the stacks have the following form:

- $a$
- $p$
- $p$
Moment 3:

\[ \begin{array}{|c|c|c|c|} \hline - & a & p & p \quad \cdots \quad f \\ \hline \end{array} \]

Here, the stacks have the following form:

\[ \begin{array}{|c|c|} \hline a & p \\ \hline \end{array} \]

Moment 4:

\[ \begin{array}{|c|c|c|c|} \hline - & a & p & p \quad \cdots \quad f \\ \hline \end{array} \]

Here, the stacks have the following form:

\[ \begin{array}{|c|c|c|} \hline p & a & p \\ \hline \end{array} \]

Moment 5:

\[ \begin{array}{|c|c|c|c|} \hline - & a & p & p \quad \cdots \quad f \\ \hline \end{array} \]

Here, the stacks have the following form:

\[ \begin{array}{|c|c|c|} \hline p & p & a \\ \hline \end{array} \]

Moment 6:

\[ \begin{array}{|c|c|c|c|} \hline - & a & p & p \quad \cdots \quad f \\ \hline \end{array} \]

Here, the stacks remain the same.
**Problem 3d-e.** Design Turing machines for computing \( a - 1 \) in unary and in binary codes. Trace both Turing machines for \( a = 5 \).

**Solution: unary case.** The rules are:

- start, \(- \to R\), moving
- moving, \(1 \to R\)
- moving, \(- \to L\), erasing
- erasing, \(1 \to -\), back
- back, \(1 \to L\)
- back, \(- \to \text{halt}\)

Tracing:

```
  - 1 1 1 1 1 - ...
    - \(\frac{1}{2}\) 1 1 1 1 - ...
    - 1 \(\frac{1}{2}\) 1 1 1 - ...
    - 1 1 \(\frac{1}{2}\) 1 1 - ...
    - 1 1 1 \(\frac{1}{2}\) 1 - ...
    - 1 1 1 1 \(\frac{1}{2}\) - ...
    - 1 1 1 1 1 - ...
    - 1 1 1 1 1 - ...
    - 1 1 1 1 1 - ...
    - 1 1 1 1 1 - ...
    - 1 1 1 1 1 - ...
```

**Solution: binary case.** The rules are:

- start, \(- \to R\), moving
- moving, \(0 \to 1\), \(R\)
- moving, \(1 \to 0\), \(L\), back
- back, \(1 \to L\)
- back, \(- \to \text{halt}\)

Here is tracing:

```
  - 1 0 1 - - ...
    - \(\frac{1}{2}\) 0 1 - - ...
    - 0 0 1 - - ...
    - 1 1 0 - - ...
    - 1 1 1 0 - - ...
```
Problem 4. Beyond Turing machines: computability

Problem 4a. Formulate Church-Turing thesis. Is it a mathematical theorem? Is it a statement about the physical world?

Solution: Church-Turing thesis states that any function that can be computed on any physical device can also be computed by a Turing machine (or, equivalently, by a Java program).

Whether this statement is true or not depends on the properties of the physical world. Thus, this statement is not a mathematical theorem, it is a statement about the physical world.
Problem 4b. Prove that the halting problem is not algorithmically solvable.

Solution: The halting problem is the problem of checking whether a given program $p$ halts on given data $d$. We can prove that it is not possible to have an algorithm $\text{haltChecker}(p,d)$ that always solves this problem by contradiction. Indeed, suppose that such an algorithm – i.e., such a Java program – exists. Then, we can build the following auxiliary Java program:

```java
public static int aux(String x)
{if(haltChecker(x,x))
  (while(true) x= x;)
else{return 0;}}
```

If $\text{aux}$ halts on $\text{aux}$, then $\text{haltChecker}(\text{aux},\text{aux})$ is true, so the program $\text{aux}$ goes into an infinite loop – and never halts. On the other hand, if $\text{aux}$ does not halt on $\text{aux}$, then $\text{haltChecker}(\text{aux},\text{aux})$ is false, so the program $\text{aux}$ returns 0 – and thus, halts. In both cases, we get a contradiction, which proves that $\text{haltChecker}$ is not possible.
Problem 4c. Not all algorithms are feasible, but, unfortunately, we do not have a perfect definition is feasibility. Give a current formal definition of feasibility and give two examples:

- an example of an algorithm’s running time which is feasible according to the current definition but not practically feasible, and
- an example of an algorithm’s running time which is practically feasible but not feasible according to the current definition.

Solution: An algorithm $A$ is called feasible if its running time $t_A(x)$ on each input $x$ is bounded by some polynomial $P(len(x))$ of the length $len(x)$ of the input: $t_A(x) \leq P(len(x))$. In other words, the algorithm is feasible if for each length $n$, the worst-case complexity $t^w_A(n) = \max\{t_A(x) : len(x) = n\}$ is bounded by a polynomial: $t^w_A(n) \leq P(n)$.

Time complexity $t^w_A(n) = 10^{200}$ is a constant – thus a polynomial, so from the viewpoint of the formal definition, it is feasible. However, this number is larger than the number of particles in the Universe, so it is clearly not practically feasible.

On the other hand, the function $\exp(10^{-21} \cdot n)$ is an exponential function and thus, grows faster than an polynomial, but even for largest realistic lengths $n$ – e.g., for $n = 10^{18}$ – the resulting value is smaller than 3 and is, thus, perfectly practically feasible.
Problem 4d. Briefly describe what is P, what is NP, what is NP-hard, and what is NP-complete. Is P equal to NP?

Solution: P is the class of all the problems that can be solved in polynomial time.

NP is the class of all the problems for which, once we have a candidate for a solution, we can check, in polynomial time, whether it is indeed a solution.

A problem is called NP-hard if every problem from the class NP can be reduced to this problem.

A problem is called NP-complete if it is NP-hard and itself belongs to the class NP.

At present, no one knows whether P is equal to NP. Most computer scientists believe that these two classes are different.
Problem 4e. Give an example of an NP-complete problem: what is given, and what we want to find.

Solution. An example of an NP-complete problem is propositional satisfiability:

- given: a propositional formula, i.e., any expression obtained from Boolean variables by using “and”, “or”, and “not”,

- find: the values of the Boolean variables that makes this formula true.
**Problem 4f.** Give definitions of a recursive (decidable) language and of a recursively enumerable (Turing-recognizable) language.

**Solution.** A language $L$ is called *decidable* if there exists an algorithm (or, equivalently, a Turing machine) that:

- given a word,
- returns “yes” or “no” depending on whether this word belongs to this language or not.

A language is called *semi-decidable, Turing-recognizable, or recursively enumerable* if there exists a Turing machine that:

- given a word $w$,
- returns “yes” if and only if the word $w$ belongs to the language $L$. 