Solution to Homework 4

Task: Prove that the following language is not regular
\{a^n b^{3n}, n = 0, 1, 2, \ldots \} = \{\Lambda, abbb, aabbbbbb, aaabbbbbbbbb, \ldots \}.

Solution. We will prove this result by contradiction. Let us assume that the language \(L\) is regular, and let us show that this assumption leads to a contradiction.

Since this language is regular, according to the Pumping Lemma, there exists an integer \(p\) such that every word from \(L\) whose length \(\text{len}(w)\) is at least \(p\) can be represented as a concatenation \(w = xyz\), where:

- \(y\) is non-empty;
- the length \(\text{len}(xy)\) does not exceed \(p\), and
- for every natural number \(i\), the word \(xy^i z \overset{\text{def}}{=} xy \ldots yz\), in which \(y\) is repeated \(i\) times, also belongs to the language \(L\).

Let us take the word \(w = a^p b^{3p} = a \ldots ab \ldots b\), in which first the letter \(a\) is repeated \(p\) times and then the letter \(b\) is repeated \(3p\) times. The length of this word is \(p + 3p = 4p > p\). So, by pumping lemma, this word can be represented as \(w = xyz\) with \(\text{len}(xy) \leq p\). The word \(w = xyz\) starts with \(xy\), and the length of \(xy\) is smaller than or equal to \(p\). Thus, \(xy\) is among the first \(p\) symbols of the word \(w\) – and these symbols are all \(a\)'s. So, the word \(y\) only has \(a\)'s.

In the original word \(w = xyz\), we had \(a\) repeated \(p\) times and \(b\) repeated \(3p\) times. When we go from the word \(w = xyz\) to the word \(xyyz\), we add \(a\)'s, and we do not add any \(b\)'s. Thus, we still have \(b\) repeated \(3p\) times, but the number of times \(a\) is repeated is now larger than \(p\). In any word from the language \(L\), we should have exactly three times more \(b\)'s than \(a\)'s. So, in the word \(xyyz\), this balance is disrupted. Thus, the word \(xyyz\) cannot be in the language \(L\).

On the other hand, by Pumping Lemma, the word \(xyyz\) must be in the language \(L\). So, we proved two opposite statements:

- that this word is not in \(L\) and
• that this word is in $L$.

This is a contradiction.

The only assumption that led to this contradiction is that $L$ is a regular language. Thus, this assumption is false, so $L$ is not regular.