Quiz 2 and Its Solution

Quiz 2. Use a general algorithm to transform the following finite automaton into the corresponding regular expression. This automaton has 2 states: 1 and 2. From each state, A leads back to the same state, while B leads to another state.

Solution. We start with the described automaton:

![Automaton Diagram]

According to the general algorithm, first we add a new start state $ns$ and a new final state $f$, and we add jumps:

- from the new start state $ns$ to the old start state, and
- from each old final state to the new final state $f$.

As a result, we get the following automaton:

![Automaton Diagram]

Then, we need to eliminate the two intermediate states 1 and 2 one by one. We can start with eliminating 1 or with eliminating 2. Let us show what happens in both cases.

First version, when we first eliminate the state 1. First, we draw all possible arrows:
Now, to find expressions to place at all these arrows, we will use the general formula
\[ R'_{i,j} = R_{i,j} \cup (R_{i,k} R_{k,k}^* R_{k,j}), \]
where \( k \) is the state that we are eliminating, i.e., in this case, the state \( k = 1 \).

By applying this formula, and by using simplification formulas described in the lecture, we get the following results:
\[
\begin{align*}
R'_{\text{ns}, 2} &= R_{\text{ns}, 2} \cup (R_{\text{ns}, 1} R_{1,1}^* R_{1,s1}) = \emptyset \cup (A A^* B) = \emptyset \cup A^* B = A^* B; \\
R'_{\text{ns}, f} &= R_{\text{ns}, f} \cup (R_{\text{ns}, 1} R_{1,1}^* R_{1,f}) = \emptyset \cup (\Lambda A^* \emptyset) = \emptyset \cup \emptyset = \emptyset; \\
R'_{2,2} &= R_{2,2} \cup (R_{2,1} R_{1,1}^* R_{1,2}) = A \cup B A^* B; \\
R'_{s,f} &= R_{2,f} \cup (R_{2,1} R_{1,1}^* R_{1,f}) = \Lambda \cup (B A^* \emptyset) = \Lambda \cup \emptyset = \Lambda.
\end{align*}
\]
Thus, the 3-state a-automaton takes the following form:
\[
\begin{align*}
A \cup B A^* B \\
\uparrow & A^* B \\
\Lambda \\
\downarrow & \emptyset \\
\Lambda \\
\downarrow & \emptyset \\
f
\end{align*}
\]
Now, all that remains to do is to go from here to the 2-state a-automaton by eliminating the remaining state 2:
The final expression is the corresponding expression for $R'_{ns,f}$:

$$R'_{ns,f} = R_{ns,f} \cup (R_{ns,2}R_{2,2}^*R_{2,f}) = \emptyset \cup (A^*B(A \cup BA^*)^*\Lambda) = \emptyset \cup A^*B(A \cup BA^*)^* = A^*B(A \cup BA^*)^*.$$  

The formula on the previous line is a regular expression corresponding to the original automaton.

**First version – answer:** $A^*B(A \cup BA^*)^*$.

**Second version, when we first eliminate the state 2.** First, we draw all possible arrows:

![Diagram of the automaton with state 2 eliminated]

Now, to find expressions to place at all these arrows, we will use the general formula

$$R'_{i,j} = R_{i,j} \cup (R_{i,k}R_{k,k}^*R_{k,j}),$$

where $k$ is the state that we are eliminating, i.e., in this case, the state $k = 2$.

By applying this formula, and by using simplification formulas described in the lecture, we get the following results:

$$R'_{ns,1} = R_{ns,1} \cup (R_{ns,2}R_{2,2}^*R_{2,s}) = \Lambda \cup (\emptyset \ldots) = \Lambda \cup \emptyset = \Lambda;$$

$$R'_{ns,f} = R_{ns,f} \cup (R_{ns,2}R_{2,2}^*R_{2,f}) = \emptyset \cup (\emptyset \ldots) = \emptyset \cup \emptyset = \emptyset;$$

$$R'_{1,1} = R_{1,1} \cup (R_{1,2}R_{2,2}^*R_{2,1}) = A \cup (BA^*B);$$

$$R'_{1,f} = R_{1,f} \cup (R_{1,2}R_{2,2}^*R_{2,f}) = \emptyset \cup (BA^*A) = \emptyset \cup BA^* = BA^*.$$  

Thus, the 3-state a-automaton takes the following form:
Now, all that remains to do is to go from here to the 2-state a-automaton by eliminating the remaining state $s$:

The final expression is the corresponding expression for $R'_{ns,f}$:

$$R'_{ns,f} = R_{ns,f} \cup (R_{ns,1}R_{1,1}^*R_{1,f}) =\emptyset \cup (\Lambda(A \cup BA^*B)^*BA^*) =\emptyset \cup (A \cup BA^*B)^*BA^* = (A \cup BA^*B)^*BA^*.$$ 

The formula in the previous line is also a regular expression corresponding to the original automaton.