1. Similarly to how we used Newton's method to design algorithms for computing square root and cubic root, design an algorithm for computing the logarithm \( x = \ln(a) \) as a solution to the equation \( e^x = a \).

Suppose we know the approximate solution \( x^{(k)} \) at step \( k \), then

\[
x^{(k+1)} = x^{(k)} + \Delta x
\]

According to Newton's method,

\[
\Delta x = -\frac{f(x^{(k)})}{f'(x^{(k)})}
\]

\[
f(x) = e^x - a = 0
\]

\[
f'(x) = e^x
\]

\[
\Delta x = -\frac{e^x - a}{e^x} = \frac{a - e^x}{e^x} = \left(\frac{a}{e^x} - 1 \right)
\]

(1) Approximate a value, for example, \( x^{(0)} = 0 \).

(2) Compute \( \Delta x \) where \( x = x^{(k)} \) \( [k+1 \text{ first time } k=0] \).

(3) \( x^{(k+1)} = x^{(k)} + \Delta x \).

(4) Repeat the step (2) to (3) until the approximation of \( x \) is very close to solution meaning the error is less than a predefined tolerance.
2. Use the algorithm for computing $1/b$ that we had in class (and that is implemented in the computers) to perform the few first steps of computing the ratio $1/1.1$.

\[
\begin{align*}
\text{Let } \frac{1}{b} &= x \quad \text{such that } x \cdot b &= 1. \\
F(x) &= x \cdot b - 1 = 0 \\
F'(x) &= b \\
\text{using Newton's method,} \\
\Delta = -\frac{F(x)}{F'(x)} = -\frac{x \cdot b - 1}{b} &= \frac{1 - x \cdot b}{b} \\
&= (1 - x \cdot b) \cdot \frac{1}{b}
\end{align*}
\]

In every step, $x$ is the approximate value for $(\frac{1}{b})$. Therefore, we can write,

\[
\Delta x = (1 - x \cdot b) \cdot x
\]

So at step $k+1$,

\[
x^{(k+1)} = x^{(k)} + (1 - x^{(k)} \cdot b) \cdot x^{(k)}
\]

In the problem, $b = 1.1$

Let $x^{(0)} = 1$,

- Iteration 1, $x^{(1)} = 0.9$
- Iteration 2, $x^{(2)} = 0.909$
- Iteration 3, $x^{(3)} = 0.909090909$

$x^{(3)}$ is very close to solution, so we can stop here.

\[
x = 0.9090909.
\]
3-6.

3. Use numerical differentiation to compute the derivative of the function \( x^2 - x \) when \( x = 1 \).

4. Use linearization technique and your estimate for the derivative to estimate the range of this function when \( x \) is in the interval \([0.9, 1.1]\).

5. Use naive interval computations to estimate the same range.

6. Use mean value form to estimate the same range.

3. By definition of differentiation,
\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}
\]

for some small value of \( h \ll 1 \), we can write,
\[
F'(x) \approx \frac{F(x+h) - F(x)}{h}
\]

Here, \( F(x) = x^2 - x \).

We want to find \( F'(x) \) at point \( x = 1 \).

Let \( h = 0.001 \).
\[
F'(x) = \frac{F(1.001) - F(1)}{0.001}
\]
\[
= 1.001
\]

If \( h = 0.00001 \),
\[
F'(x) = \frac{F(1.00001) - F(1)}{0.00001} = 1.00001
\]

Analytical solution: \( F'(1) = 1 \), which is very close to numerical differentiation.
4. Here given that

\[ y = f(x) = x^2 - x \quad \text{where} \]

\[ x \in \left[0.9, 1.1\right] \]

\[ \bar{x} = \frac{x + \bar{x}}{2} = \frac{0.9 + 1.1}{2} = 1 \]

\[ \Delta x = \frac{x - x}{2} = \frac{1.1 - 0.9}{2} = 0.1 \]

\[ y' = f'(\bar{x}) = 2\bar{x} - 1 = 0 \]

We know,

\[ \Delta = \left| \frac{\partial f}{\partial x} \right| \Delta x \]

\[ = \left| (2\bar{x} - 1) \right| \Delta x \]

\[ = \left| (2 \cdot 1 - 1) \right| \cdot 0.1 \]

\[ = 1 \cdot 0.1 \]

\[ = 0.1 \]

The range of \( y = \left[ 0 - 0.1, 0 + 0.1 \right] \)

\[ = \left[ -0.1, 0.1 \right] \]

5. \( f(x) = x^2 - x \quad \text{where} \quad x \in \left[0.9, 1.1\right] \)

\[ = \left[0.9, 1.1\right]^2 - \left[0.9, 1.1\right] \]

\[ = \left[0.81, 1.21\right] - \left[0.9, 1.1\right] \]

\[ = \left[-0.19, 0.11\right] \]
6. \( f(x) = x^2 - x \) where \( x \in [0.9, 1.1] \)
\[ \Delta x = 0.1, \quad \Delta x = 1 \]
\[ \frac{df}{dx} = 2x - 1 \]
\[ y = f(\tilde{x}) = 1^2 - 1 = 0 \]

According to mean-value form
\[ f(x) \in \tilde{y} + \left[ \frac{df}{dx} \right] \cdot [-\Delta x, \Delta x] \]
\[ = 0 + [2x - 1] \cdot [-\Delta x, \Delta x] \]
\[ = \frac{2x - 1}{2} \cdot [0.9, 1.1] - 1 \]
\[ = [1.8, 2.2] - 1 \]
\[ = [0.8, 1.2] \]

\[ f(x) \in [0.8, 1.2] \cdot [-0.1, 0.1] \]
\[ = [-0.12, 0.12] \]
7. Use Newton's method to solve the following system of non-linear equations:

\[ x_1 \times x_2 = 3, \ x_1 + x_2 = 4. \]

Start with the first approximation \( x_1 = 1 \) and \( x_2 = 2 \). One iteration is good enough.

**Given**

\[ f_1(x_1, x_2) = x_1 \times x_2 - 3 = 0 \]
\[ f_2(x_1, x_2) = x_1 + x_2 - 4 = 0 \quad \text{and} \quad x_1^{(0)} = 1 \quad \text{and} \quad x_2^{(0)} = 2 \]

According to Newton's method for systems of non-linear equations, we can write:

\[
\frac{\partial f_1}{\partial x_1} \Delta x_1 + \frac{\partial f_1}{\partial x_2} \Delta x_2 = -f_1(x_1^{(0)}, x_2^{(0)})
\]
\[
\frac{\partial f_2}{\partial x_1} \Delta x_1 + \frac{\partial f_2}{\partial x_2} \Delta x_2 = -f_2(x_1^{(0)}, x_2^{(0)})
\]

\[
\frac{\partial f_1}{\partial x_1} = x_2 \quad \frac{\partial f_2}{\partial x_1} = 1
\]
\[
\frac{\partial f_1}{\partial x_2} = x_1 \quad \frac{\partial f_2}{\partial x_2} = 1
\]

So,
\[
f_1(x_1^{(0)}, x_2^{(0)}) = 1 \times 2 - 3 = -1
\]
\[
f_2(x_1^{(0)}, x_2^{(0)}) = 1 + 2 - 4 = -1
\]

\[
2 \Delta x_1 + \Delta x_2 = 1
\]
\[
\Delta x_1 + \Delta x_2 = 1
\]

\[
\Delta x_1 = 0
\]
\[
\text{and} \quad \Delta x_2 = 1.
\]

Thus,
\[
x_1^{(1)} = 1 + 0 = 1
\]
\[
x_2^{(1)} = 2 + 1 = 3
\]

**Solution**, \( x_1 = 1 \) and \( x_2 = 3 \).
8. Find the point closest the origin on the line \( x_1 - x_2 = 1 \). In other words, find the values \( x_1 \) and \( x_2 \) for which the sum \((x_1)^2 + (x_2)^2\) attains the smallest possible value under the constraint \( x_1 - x_2 = 1 \).

\[
\begin{align*}
\mathcal{f}(x) &= (x_1)^2 + (x_2)^2 \\
\mathcal{g}(x) &= x_1 - x_2 - 1 = 0
\end{align*}
\]

Using Lagrange multiplier,

\[
\mathcal{L}(x) = (x_1)^2 + (x_2)^2 + \lambda (x_1 - x_2 - 1)
\]

\[
\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 + \lambda = 0
\]

\[
\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - \lambda = 0
\]

\[
\begin{align*}
x_1 &= -\frac{x}{2} \\
\lambda &= \frac{x_1}{2} = -x_1
\end{align*}
\]

From \( \mathcal{g}(x) = 0 \), putting the value of \( x_1 \) and \( x_2 \),

\[
\begin{align*}
6x_1 &= 1 \\
x_1 &= \frac{1}{2}
\end{align*}
\]

and \( -2x_2 = 1 \)

\[
\begin{align*}
x_2 &= -\frac{1}{2}
\end{align*}
\]

\[
(x_1, x_2) = \left( \frac{1}{2}, -\frac{1}{2} \right)
\]
9. Explain what is k-anonymity, and why it is important. If k increases, will we get more or less privacy protection? Explain your answer.

Let a desired statistical characteristics for a dataset, \( C(x^{(1)}, \ldots, x^{(n)}, \ldots, x^{(N)}, \ldots, x^{(N)}) \).

To protect privacy of the dataset, we can introduce interval instead of exact values. Assuming that the dataset distribute over a hyperspace, we can make cells for every characteristic to introduce interval.

k-anonymity is the condition where in every cell for different characteristics, there should be at least k different people after all queries.

This is the primary important concept in data privacy. Assume, from a people record dataset, we erase name of people, still it's not private because if we know the scope/all people of the dataset, and allows to query for n-1 people, we can easily identify any private information of a particular person. k-anonymity solves this problem.

If k-increases, we will get more privacy because for every query we will find at least k people, which is difficult to identify a particular person.