For loops are not enough—we want to prove:

There exists a computable function $f$ which is not P.R. search for something—→ is while loop.

2 proofs:
1) with more detail, but $f(n)$ will be useless
2) with less detail, but $f(n)$ will be useful

We need some auxiliary notion for the proof:

the code $\delta$ is a P.R. function.

(i) A P.R. function can be described as an expression $F^3 \overset{\gamma}{\rightarrow} 0$

\[
\text{latex}
\]

(ii) $P(R(\sigma, \pi \wedge 3-1 \wedge \text{cire} \sigma))

\text{ASCII}

(iii) $101101\ldots$
append 1 in front because we're dealing with non-negative integers.

$101101\ldots$ → interpret it as an integer, this is called the code $\delta$ of a P.R. function.
Lemma:
There exists an algorithm that, given a natural number \( c \),
* checks whether \( c \) is a code of a p.r. fnc
* if yes, returns the executable file computing this fnc
this file will be denoted \( f_c \).

Let us construct a function \( f(c) \) about which we will prove that it is computable but not p.r.
\[
f(c) = \begin{cases} 
\hat{f}_c(c) + 1 & \text{if } c \text{ is a code of a p.r. fnc} \\
0 & \text{otherwise} 
\end{cases}
\]
Let's prove that \( f(n) \) is computable.

```
\[ \begin{array}{c}
\text{is } c \text{ a code of a p.r. fnc} \\
\text{Yes} \\
\text{No} \\
\text{Return 0}
\end{array} \quad \begin{array}{c}
\text{yes} \\
\text{generate } f_c \\
\text{apply } f_c \text{ to } e \\
\text{add 1} \\
\text{return } f_c(c) + 1
\end{array} \]
```
To prove the theorem, we now need to show that $f(c)$ is not p.r.

We will prove it by contradiction, we assume that $f(c)$ is p.r., and we will get a contradiction.

Let $c_0$ be a code of p.r. fnc $f(c)$.

\[
\forall n \in \mathbb{N} \quad f_{c_0}(n) = f(n)
\]

What is the value of $f_{c_0}(c_0)$?

This is true for all $n$, in particular for $c = c_0$.

$f_{c_0}(c_0) = f(c_0)$.

On the other hand, by def of $f(c)$,

\[
f(c) = f_{c_0}(c_0) + 1
\]

$f_{c_0}(c_0) = f_{c_0}(c_0) + 1$

$0 = 1$, which is a contradiction.

This proves that $f(c)$ is not p.r.

\[
f(c) = \begin{cases} f_{c_0}(c) + 1 & \text{if } c \text{ is a code of a p.r fnc } c_0 \\ 0 & \text{otherwise} \end{cases}
\]

Informal explanation based on a toy example:

- $f_0(n) = 0$
  - is not a code
- $f_2(n) = n + 1$
- $f_3(n) = n^2$
- $f_4(n) = 2 \times n$
Let's form a table

<table>
<thead>
<tr>
<th>$f^n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>$P_4$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>$f^n$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

We are using only the diagonal element, so it's called diagonal construction.

George Cantor proves that the set of real numbers cannot be countable.

\[ n+1 \]

\[ a + b = a + \ldots + a \text{ } b \text{ times} \]

\[ a \times b = a + a + \ldots + a \text{ } b \text{ times} \]

\[ a^b = a \times a \times \ldots \times a \text{ } b \text{ times} \]

\[ b^a = a \times a \times \ldots \times a \text{ } b \text{ times} \text{ Archimedes} \]
0-th order \[ f_0(a, b) = a + 1 \]
1-st order \[ f_1(a, b) = a + b \]
2-nd order \[ f_2(a, b) = a \cdot b \]
3-rd order \[ f_3(a, b) = a^b \]
4-th order \[ f_4(a, b) = f_3(a, f_3(a, \ldots)) = a \text{ repeated } b \text{ times} \]

\[ f_4(a, 0) = 1 \]
\[ f_4(a, m+1) = f_3(a, f_4(a, m)) \]
\[ f_{k+1}(a, m+1) = f_k(a, f_k(a, m)) \]

Each of these operations is p.r.

Ackermann describes the func
\[ A(m) = f_4(n, n) \]

Thm: \[ A(n) \text{ is not p.r.} \]

Hint: \[ A(0) = f_4(0, 0) = 0 + 1 = 1 \]
\[ A(1) = f_4(1, 1) = 1 + 1 = 2 \]
\[ A(2) = f_4(2, 2) = 2 \cdot 2 = 4 \]
\[ A(3) = f_4(3, 3) = 3 \cdot 3 = 27 \]
\[ A(4) = f_4(4, 4) = 4 \cdot 4 = 4^4 \]

\[ 2^{512} = (2^{10})^{512} = 2^{5120} \]

\[ 4^{4} = 4 \cdot 1536 \]