Solution to Homework 41

**Homework 41.** On the example of the negation function \( f(x) = \neg x \), trace, step by step, how Deutsch-Josza algorithm will conclude that \( f(0) \neq f(1) \) while applying \( f \) only once.

**Solution.** The Deutsch-Josza algorithm consists of the following steps:

- we start with the state \(|0,1\rangle = |0\rangle \otimes |1\rangle\); 
- we apply the Hadamard transformation \( H \) to both bits, i.e., the transformation for which 
  \[ H(|0\rangle) = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle; \quad H(|1\rangle) = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle; \]
- then, we apply the function \( f \), i.e., apply the transformation 
  \[ f(|x, y\rangle) = |x, y \oplus f(x)\rangle, \]
  where \( a \oplus b \) means addition modulo 2 or, equivalently, exclusive “or”:
  
  \[ 0 \oplus 0 = 0; \quad 0 \oplus 1 = 1 \oplus 0 = 1; \quad 1 \oplus 1 = 0; \]
- after that, we again apply the Hadamard transformation to both bits;
- finally, we measure the first bit of the resulting 2-bit state: 
  - if the first bit is 0, we conclude that the function \( f \) is constant;
  - if the first bit is 1, we conclude that the function \( f \) is not constant.

According to the handout, after applying the Hadamard transformation \( H \) to both bits of the state \(|0,1\rangle = |0\rangle \otimes |1\rangle\), we get the state 

\[ H(|0\rangle) \otimes H(|1\rangle) = \frac{1}{2}|0,0\rangle - \frac{1}{2}|0,1\rangle + \frac{1}{2}|1,0\rangle - \frac{1}{2}|1,1\rangle. \quad (1) \]

When we apply the function \( f \), we get the following: 

\[ f(|0,0\rangle) = |0,1\rangle, \quad f(|0,1\rangle) = |0,0\rangle, \quad f(|1,0\rangle) = |1,0\rangle, \quad f(|1,1\rangle) = |1,1\rangle. \]

Thus, the state \((1)\) gets transformed into 

\[ f(H(|0\rangle) \otimes H(|1\rangle)) = \frac{1}{2}|0,1\rangle - \frac{1}{2}|0,0\rangle + \frac{1}{2}|1,0\rangle - \frac{1}{2}|1,1\rangle = \]

1
\[-\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |0\rangle - \frac{1}{2}|1\rangle \otimes |1\rangle.\]

The first two terms have a common factor $|0\rangle$, the third and the fourth one have a common factor $|1\rangle$, so we have

\[f(H(|0\rangle) \otimes H(|1\rangle)) = -\frac{1}{\sqrt{2}}|0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).\]

This expression can be equivalently reformulated as

\[f(H(|0\rangle) \otimes H(|1\rangle)) = -\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right).\]

For both bits, what we have is $H|1\rangle$:

\[f(H(|0\rangle) \otimes H(|1\rangle)) = -H(|1\rangle) \otimes H(|1\rangle).\]

It is known that when we apply the Hadamard transformation twice, we get back the original state. In particular, $H(H(|1\rangle)) = |1\rangle$. Thus, when we apply the Hadamard transformation to both bits once again, we get the state

\[-|1\rangle \otimes |1\rangle.\]

Measuring the value of the first bit, we get the value 1 with probability $| -1 |^2 = 1$. Thus, we can indeed conclude that $f(0) \neq f(1)$ – and we called the function $f$ only once.