Solution to Problem 12

**Problem.** In class, we described a Turing machine that computes \( g(n) = n + 1 \). In Homework 9, you designed a Turing machine that computes a function \( f(n) \) which is equal to \( n - 2 \) when \( n \geq 2 \) and to 0 when \( n = 0 \).

In class, we described the general algorithm for designing a Turing machine that computes the composition of two functions. The assignment is to use this general algorithm to design a Turing machine that computes the composition \( g(f(n)) \). Trace, step-by-step, on an example, how your Turing machine works. For example, you can take as input \( n = 2 \).

*Reminder:* The Turing machine for computing \( g(n) = n + 1 \) for a unary input \( n \) is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we replace this blank space with 1 and go back.

This machine has the following rules:

- start, \( \rightarrow \) R, working
  (we start going to the right)
- working, 1 \( \rightarrow \) R
  (we see 1, so we continue going),
- working, \( \rightarrow \) 1, L, back
  (we see a blank space, so we replace it with 1 and start going back)
- back, \( \rightarrow \) L
  (while we see 1s, we continue going back)
- back, \( \rightarrow \) halt
  (once we reach the very first cell, we stop).

The Turing machine for computing \( f(n) \) is based on the following idea:

- we go step-by-step until we find the first blank space,
- then, we go back, replace the last two 1s with blanks, and go back all the way.

We need to take special care of the case when \( n = 0 \).
Solution. The resulting Turing machine takes the following form:

- start, \( \rightarrow \) R, moving\(_1\)
- moving\(_1\), 1 \( \rightarrow \) R
- moving\(_1\), \( \rightarrow \) erasing\(_{1st1}\)
- erasing\(_{1st1}\), 1 \( \rightarrow \), L, erasing\(_{2nd1}\)
- erasing\(_{2nd1}\), 1 \( \rightarrow \), L, back\(_1\)
- back\(_1\), 1 \( \rightarrow \) L
- back\(_1\), \( \rightarrow \) start\(_2\)
- erasing\(_{1st1}\), \( \rightarrow \) start\(_2\)
- erasing\(_{2nd1}\), \( \rightarrow \) start\(_2\)
- start\(_2\), \( \rightarrow \) R, working\(_2\)
- working\(_2\), 1 \( \rightarrow \) R
- working\(_2\), \( \rightarrow \) 1, L, back\(_2\)
- back\(_2\), \( \rightarrow \) L
- back\(_2\), \( \rightarrow \) halt

Let us trace it for \( n = 2 \);

\[
\begin{array}{c|c|c|c|c|c|c}
_& 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
_& _ & 1 & 1 & \cdots \\
\hline
\end{array}
\]

start

moving\(_1\)

moving\(_1\)

moving\(_1\)

erasing\(_{1st1}\)

erasing\(_{2nd1}\)

back\(_1\)

start\(_2\)

working\(_2\)

back\(_2\)

halt