Solution to Problem 15

**Problem.** Sketch an example of a Turing machine for implementing μ-recursion, the way we did it in class, on the example of a function \(\mu m. (m = a)\).

**Solution.** The given function is a particular case of a general \(\mu\)-recursion expression

\[
f(n_1, \ldots, n_k) = \mu m. P(n_1, \ldots, n_k, m)
\]

Corresponding to \(k = 1\) and \(P(n_1, m) \Leftrightarrow m = n_1\).

Suppose that we have a Turing machine for computing the equality relation \(P(n_1, m)\). According to the general algorithm described in the lecture, we start with the state

\[
\underline{-} n_1 \underline{-} \ldots \text{ start}
\]

and we want to end up in the state

\[
\underline{-} \text{f(n1)} \underline{-} \ldots \text{ halt}
\]

Let us show how this can be done. First, we add 0 after the input, copy the whole tuple \((n_1, 0)\), and move the head before the second copy of \(n_1\):

\[
\underline{-} n_1 \underline{-} 0 \underline{-} \ldots
\]

Then, we apply the Turing machine computing the function \(P(n_1, 0)\). As a result, we get the following state:

\[
\underline{-} n_1 \underline{-} 0 \underline{-} P(n_1, 0) \underline{-} \ldots
\]

If \(P(n_1, 0) = 0\) (i.e., if the property \(P(n_1, m)\) is false), then we increase 0 by 1, copy the tuple \((n_1, 1)\):

\[
\underline{-} n_1 \underline{-} 1 \underline{-} \ldots
\]

and again apply the Turing machine for computing \(P(n_1, m)\), resulting in:

\[
\underline{-} n_1 \underline{-} 1 \underline{-} P(n_1, 1) \underline{-} \ldots
\]

In general, at each iteration, we start with the state

\[
\underline{-} n_1 \underline{-} m \underline{-} P(n_1, m) \underline{-} \ldots
\]

If \(P(n_1, m) = 0\) (i.e., to “false”), then we increase \(m\) by 1, copy the tuple \((n_1, m + 1)\):

\[
\underline{-} n_1 \underline{-} m + 1 \underline{-} \ldots
\]
and again apply the Turing machine for computing $P(n_1, m + 1)$, resulting in:

$$- n_1 - m + 1 - P(n_1, m + 1) - \ldots$$

etc.

This continues until we get the first value $m$ for which $P(n_1, m) = 1$ (i.e., “true”). In this case, we get the state

$$- n_1 - m - 1 - \ldots$$

Here, the desired value $m$ is 2-nd out of 3, so it can be found if we apply the corresponding projection $\pi_3^2$, resulting in:

$$- m - \ldots \text{ halt}$$

where $m = f(n_1) = \mu m. P(n_1, m)$.

This is exactly what we wanted.