Solution to Problem 1

**Problem 1.** Prove that the function computing the sum $1 + 2 + \ldots + n$ is primitive recursive. This proof should follow the same pattern that we used in class to prove that addition and multiplication are primitive recursive:

- You start with a 3-dot expression.
- First you write a for-loop corresponding to this function
- Then you describe this for-loop in mathematical terms
- Then, to prepare for a match with the general expression for primitive recursion, you rename the function to $f$ and the parameters to $n_1, \ldots, n_k, m$
- Then you write down the general expression of primitive recursion for the corresponding $k$
- Then you match: find $g$ and $h$ for which the specific case of primitive recursion will be exactly the functions corresponding to initialization and to what is happening inside the loop
- Then, you get a final expression for the function $1 + 2 + \ldots + n$ that proves that this function is primitive recursive, i.e., that it can be formed from $0, \pi_k, \sigma$ by composition and primitive recursion.

**Solution.** Here is the for-loop for computing the desired expression:

```c
int sum = 0;
for (int i = 1; i <= m; i++){
    sum = c + i;
}
```

Let us now describe this for-loop in mathematical terms:

$$
sum(0) = 0;
sum(m+1) = sum(m) + (m+1).
$$

To prepare for the match, we rename the function to $h$ (here, there are no other parameters to rename):

$$
h(0) = 0;
$$
\[ h(m + 1) = h(m) + (m + 1). \]

Here, we are defining a function of 1 variable. In general, primitive recursion defines a function of \( k + 1 \) variables. Here, \( k + 1 = 1 \), so \( k = 0 \), and the general expression for primitive recursion takes the following form:

\[
\begin{align*}
  h(0) &= f; \\
  h(m + 1) &= g(m, h(m)).
\end{align*}
\]

To match with the above description, we need to take \( f = 0 \) and \( g(m, h) = h + (m + 1) \), i.e., \( g = \text{add}(\pi_2^2, \sigma \circ \pi_1^2) \).

Thus, the desired expression for our function is

\[
\text{sum} = PR(0, \text{add}(\pi_2^2, \sigma \circ \pi_1^2))
\]