Solution to Homework 37

Problem. What class of polynomial hierarchy contains $\Sigma^P_{\Pi^P_1}$? Explain your answer.

Solution. For each oracle $A$, the class $\Sigma^P_A$ is described by formulas $F$ with 4 quantifiers starting with the existential quantifier in which the main property is feasible with respect to $A$:

$$F \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4 C^A(x_1, x_2, x_3, x_4, x).$$

(1)

Here, the fact that $C^A$ is feasible with respect to the corresponding oracle $A = \Pi^P_1$ means that this property, in its turn, has the form

$$C^A(x_1, x_2, x_3, x_4, x) = C^{\Pi^P_1}(x_1, x_2, x_3, x_4, x) \equiv \forall x_5 C(x_1, x_2, x_3, x_4, x_5, x),$$

(2)

where the property $C(x_1, x_2, x, x_3, x_4, x_5, x)$ is actually feasible.

Substituting the formula (2) into the expression (1), we get

$$F \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4 \forall x_5 C(x_1, x_2, x_3, x_4, x_5, x).$$

(3)

As mentioned in the lecture, two consequent quantifiers – such as $\forall x_4 \forall x_5$ – is equivalent to $\forall(x_4, x_5)$, and can thus be compressed into a single quantifier $\forall x_4'$ – since, as we have learned earlier, there is a feasible correspondence between natural numbers and pairs of natural numbers. In this correspondence, each code $n$ of the pair can be transformed back into the pair $(\pi_1(n), \pi_2(n))$.

Thus, the formula (3) takes the simplified form

$$F \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4' C(x_1, x_2, x_3, x_4', \pi_1(x_4'), \pi_2(x_4'), x).$$

In this expression, we have 4 quantifiers, the first of which is the existential quantifier. Thus, this formula belongs to the class $\Sigma^P_4$, i.e.:

$$\Sigma^P_{\Pi^P_1} \subseteq \Sigma^P_4.$$