Solution to Problem 8

**Task.** Check and, if possible, use monotonicity to estimate the ranges of the functions from Problems 1 and 2.

**Problem 1.** Here, $f'(x) = 4 - 2x$. Thus, the interval estimate of the range of this derivative when $x \in [0, 6]$ is

$$f'([0, 6]) = 4 - 2 \cdot [0, 6] = [4, 4] - [2, 2] \cdot [0, 6] = [4, 4] - [0, 12] = [4, 12] = [4 - 4, 4 - 0] = [-8, 4].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic.

**Problem 2.** Here,

$$\frac{\partial f}{\partial x_1} = 4 - x_2,$$

so the range of this derivative is

$$4 - [-1, 6] = [4, 4] - [-1, 6] = [4 - 6, 4 - (-1)] = [-2, 5].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic with respect to $x_1$.

For the derivative with respect to $x_2$, we have

$$\frac{\partial f}{\partial x_2} = -2 - x_1,$$

so the range of this derivative is


All the values from this interval are negative, so the function $f(x_1, x_2)$ is decreasing with respect to $x_2$. Thus:

- the largest possible value of the function $f(x_1, x_2)$ is attained when the value $x_2$ is the smallest, and
- the smallest possible value of the function $f(x_1, x_2)$ is attained when the value $x_2$ is the largest.
**Estimating the largest value of the function.** The largest value is attained when the value $x_2$ is the smallest, i.e., when $x_2 = -1$. In this case,

$$f(x_1, x_2) = (2 + x_1) \cdot (4 - (-1)) = 5 \cdot (2 + x_1).$$

The derivative of this expression is $5 > 0$, so this expression is increasing. Thus, its largest value is attained when the value $x_1$ is the largest, i.e., when $x_1 = 6$. For this value $x_1$, this expression is equal to

$$5 \cdot (2 + 6) = 5 \cdot 8 = 40.$$

Thus, the largest possible value of the function $f(x_1, x_2)$ is 40.

**Estimating the smallest value of the function.** The smallest possible value of the function $f(x_1, x_2)$ is attained when the value $x_2$ is the largest, i.e., when $x_2 = 6$. In this case,

$$f(x_1, x_2) = (2 + x_1) \cdot (4 - 6) = (-2) \cdot (2 + x_1).$$

The derivative of this expression is $-2 < 0$, so this expression is decreasing. Thus, its smallest value is attained when the value $x_1$ is the largest, i.e., when $x_1 = 6$. For this value $x_1$, this expression is equal to

$$(−2) \cdot (2 + 6) = (−2) \cdot 8 = −16.$$

Thus, the smallest possible value of the function $f(x_1, x_2)$ is $−16$.

**Resulting estimate for the range.** Thus, the range of the function $f(x_1, x_2)$ is $[-16, 40]$.

It should be mentioned that this is the exactly what we got when we estimate calculus to estimate this range.