Solution to Problem 9

**Task.** Use monotonicity and one iteration of bisection to estimate the ranges of the functions from Problems 1 and 2.

**Solution.** For Problem 2, we already got the exact range due to monotonicity, so it is sufficient to consider Problem 1. For this problem, bisection means that instead of the original interval $[0, 6]$, we consider two subintervals $[0, 3]$ and $[3, 6]$, and take the union of the resulting two ranges.

**Interval** $[0, 3]$. For this interval, the range of the derivative $f'(x_1) = 4 - 2x_1$ is equal to

$$4 - 2 \cdot [0, 3] = 4 - [0, 6] = [-2, 4].$$

This range includes both positive and negative values, so we cannot conclude that the function is monotonic. Straightforward interval computations lead to

$$(2 + [0, 3]) \cdot (6 - [0, 3]) = [2, 5] \cdot [3, 6] = [6, 30].$$

**Interval** $[3, 6]$. For this interval, the range of the derivative is equal to

$$4 - 2 \cdot [3, 6] = 4 - [6, 12] = [-8, -2].$$

All the values in this range are negative, so the function is decreasing. Thus:

- its largest value is attained when $x_1$ is the smallest $x_1 = 3$, this value $f(3)$ we have already computed, it is $f(3) = 15$; and
- its smallest value is attained when $x_1$ is the largest $x_1 = 6$, this value $f(6)$ we have already computed, it is $f(6) = 0$.

Thus, for this interval, we get the range $[0, 15]$.

**The resulting range.** The resulting range is estimated as the union

$$[6, 30] \cup [0, 15]$$

of the ranges estimated for subintervals. This union is equal to

$$[\min(6, 0), \max(30, 15)] = [0, 30].$$