How Calculus Helps Find the Range:
General Case

Problem: reminder.

- We are given a function $y = f(x_1, \ldots, x_n)$ and ranges $[\underline{x}_i, \overline{x}_i]$, $i = 1, \ldots, n$.

- We want to find the range $[\underline{y}, \overline{y}] = \{ f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i] \text{ for all } i \}$

  of all possible values $y = f(x_1, \ldots, x_n)$ when $x_i \in [\underline{x}_i, \overline{x}_i]$ for all $i$.

Idea. In this case,

$$\underline{y} = \min_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n); \quad \overline{y} = \max_{x_i \in [\underline{x}_i, \overline{x}_i]} f(x_1, \ldots, x_n).$$

When a function attains maximum or minimum, then if we only consider its dependence on each of the variables, we only get minimum or maximum. Thus, for each variable $i$, either $x_i = \underline{x}_i$, or $x_i = \overline{x}_i$, or $\frac{\partial f}{\partial x_i} = 0$.

Resulting algorithm. We consider all possible combinations $(x_1, \ldots, x_n)$ for which, for each $i$, we have:

- either $x_i = \underline{x}_i$,

- or $x_i = \overline{x}_i$,

- or $\frac{\partial f}{\partial x_i} = 0$ and $x_i \in [\underline{x}_i, \overline{x}_i]$.

For each such combination, we compute the value $f(x_1, \ldots, x_n)$. The smallest of these values of the function is $\underline{y}$, the largest is $\overline{y}$.

Caution. If for each variable, we have 3 options, this means that we have to consider $3^n$ possible combinations. For large $n$, this is not feasible.

Example. Let us compute the range of a function $f(x_1, x_2) = x_1^2 - x_1 \cdot x_2 + x_2^2$ when $x_1$ is in the interval $[-1, 1]$ and $x_2$ is in the interval $[-1, 1]$.

Here, the derivative with respect to $x_1$ is equal to $2x_1 - x_2$. So, for $x_1$, we need to consider three cases:
• when \( x_1 = -1 \),
• when \( x_1 = 1 \), and
• when \( \frac{\partial f}{\partial x_1} = 0 \), i.e., when \( 2x_1 - x_2 = 0 \).

The derivative with respect to \( x_2 \) is equal to \( -x_1 + 2x_2 \). So, for \( x_2 \), we need to consider two cases:
• when \( x_2 = -1 \),
• when \( x_2 = 1 \), and
• when \( \frac{\partial f}{\partial x_2} = 0 \), i.e., when \( -x_1 + 2x_2 = 0 \).

First, let us consider cases when \( x_1 = -1 \).
• When \( x_1 = -1 \) and \( x_2 = -1 \), we get
  \[
  f(x_1, x_2) = (-1)^2 - (-1) \cdot (-1) + (-1)^2 = 1 - 1 + 1 = 1.
  \]
• When \( x_1 = -1 \) and \( x_2 = 1 \), we get
  \[
  f(x_1, x_2) = (-1)^2 - (-1) \cdot 1 + 1^2 = 1 + 1 + 1 = 3.
  \]
• When \( x_1 = -1 \) and \( -x_1 + 2x_2 = 0 \), then \( x_2 = -0.5 \), so
  \[
  f(x_1, x_2) = (-1)^2 - (-1) \cdot (-0.5) + (-0.5)^2 = 1 - 0.5 + 0.25 = 0.75.
  \]

Second, we consider cases when \( x_1 = 1 \):
• When \( x_1 = 1 \) and \( x_2 = -1 \), we get
  \[
  f(x_1, x_2) = 1^2 - 1 \cdot (-1) + (-1)^2 = 1 + 1 + 1 = 3.
  \]
• When \( x_1 = 1 \) and \( x_2 = 1 \), we get
  \[
  f(x_1, x_2) = 1^2 - 1 \cdot 1 + 1^2 = 1 - 1 + 1 = 1.
  \]
• When \( x_1 = 1 \) and \( -x_1 + 2x_2 = 0 \), then \( x_2 = 0.5 \), so
  \[
  f(x_1, x_2) = 1^2 - 1 \cdot 0.5 + 0.5^2 = 1 - 0.5 + 0.25 = 0.75.
  \]

Finally, when \( \frac{\partial f}{\partial x_1} = 0 \), i.e., when \( 2x_1 - x_2 = 0 \), then:
• When \( 2x_1 - x_2 = 0 \) and \( x_2 = -1 \), we get \( x_1 = -0.5 \), so
  \[
  f(x_1, x_2) = (-0.5)^2 - (-0.5) \cdot (-1) + (-1)^2 = 0.25 - 0.5 + 1 = 0.75.
  \]
• When $2x_1 - x_2 = 0$ and $x_2 = 1$, we get $x_1 = 0.5$, so
\[
f(x_1, x_2) = 0.5^2 - 0.5 \cdot 1 + 1^2 = 0.25 - 0.5 + 1 = 0.75.
\]

• When $2x_1 - x_2 = 0$ and $-x_1 + 2x_2 = 0$, then substituting $x_2 = 2x_1$ from
the first equation into the second equation, we get $-x_1 + 4x_1 = 0$, i.e.,
$3x_1 = 0$ and $x_1 = 0$. Thus, $x_2 - 2x_1 = 0$, and we have
\[
f(x_1, x_2) = 0^2 - 0 \cdot 0 + 0^2 = 0 - 0 + 0 = 0.
\]

So, we get the following values:

<table>
<thead>
<tr>
<th></th>
<th>$x_1 = -1$</th>
<th>$x_1 = 1$</th>
<th>$\frac{\partial f}{\partial x_1} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2 = -1$</td>
<td>1</td>
<td>3</td>
<td>0.75</td>
</tr>
<tr>
<td>$x_2 = 1$</td>
<td>3</td>
<td>1</td>
<td>0.75</td>
</tr>
<tr>
<td>$\frac{\partial f}{\partial x_2} = 0$</td>
<td>0.75</td>
<td>0.75</td>
<td>0</td>
</tr>
</tbody>
</table>

The smallest of these values is 0, the largest is 3, so the range is $[0, 3]$. 