Interval Computations

* Monotonic Case
* Linearized Case
* Numerical diff technique
* Cauchy-based method (w/ derivatives)

Reliability and trust

* General idea: Monte-Carlo est
* Stochastic: expression Z ~ N(1,1)
* Re-scaling Monte-Carlo

Possible dependencies

* Analytical f-ig for simple case
  \[ P(A \cap B) = P(A) \cdot P(B) \]
* Shortest-path alg for trust (w/oout derivation)

Intervals as a way to provide privacy

\[ a_1 \in [10, 20] \]
\[ a_2 \in [20, 30] \]

\[ M = \frac{a_1 + a_2}{2} \]
\[ U = \frac{a_1^2}{2} + \frac{a_2^2}{2} - \left( \frac{a_1 + a_2}{2} \right)^2 \]

\[ F(x_1, x_2) = \frac{x_1 + x_2}{2} \]
\[ \Delta = \sum_{i=1}^{n} \left| \frac{\partial F}{\partial x_i} \right| \Delta_i = \frac{1}{2} \cdot \Delta + \frac{1}{2} \cdot \Delta_2 = 5 \]

\[ M = [15, 25] \]

For the Variance

\[ \frac{15^2}{2} + \frac{25^2}{2} - 20^2 = \frac{225}{2} + \frac{625}{2} - 400 = \frac{425}{2} - 400 = 25 \]

\[ f(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_2^2}{2} - \left( \frac{x_1 + x_2}{2} \right)^2 \]

\[ \frac{\partial f}{\partial x_1} = \frac{x_1}{2} - 2 \left( \frac{x_1 + x_2}{2} \right) (\frac{1}{2}) = \frac{x_1 - x_2}{2} = -5 \]
\[ \frac{\partial f}{\partial x_2} = \frac{x_2 - x_1}{2}, \quad \Delta = 1 - 5 \cdot 5 + 15 \cdot 5 = 50 \]
Bisection

\[ f(x) = 2x - 3 = 0 \]
\[ x \in [0, 4] \]

\[ \begin{array}{c|c|c}
\text{lower} = 0 & \text{lower} = 1 & \text{lower} = 2 \\
\text{upper} = 4 & \text{upper} = 2 & \text{upper} = 2 \\
\text{mid} = 2 & \text{mid} = 1 & \text{mid} = 3/2 \\
\end{array} \]

Probabilities: We have two 2 events, describe the range.

\[ A \]
\[ PA = 0.8 \]
\[ PB = 0.7 \]

\[ \begin{array}{l}
\Rightarrow \left[ \max(\text{PA} + \text{PB} - 1, 0), \min(\text{PA}, \text{PB}) \right] \\
[0.8, 0.7] \\
\left[ 0.5, 0.7 \right] \\
\end{array} \]

\[ \begin{array}{l}
P(\text{A} \lor \text{B}) = P(7 (\neg \text{A} \lor \neg \text{B})) \\
[\max(\text{PA}, \text{PB}), \min(\text{PA} + \text{PB}, 1)] \\
[0.8, 1] \\
\end{array} \]
Shortest Path

\[ \Delta \]

\[ \begin{array}{c}
A \\
0.5 \\
\end{array} \]

\[ \begin{array}{c}
C \\
0.9 \\
\end{array} \]

\[ \begin{array}{c}
E \\
0.2 \\
\end{array} \]

\[ \begin{array}{c}
D \\
0.3 \\
\end{array} \]

1 - 0.4 = 0.6

Re-scaling (Monte Carlo)

\[ p_1 = 1 - \Delta \theta_1, \quad p_2 = 1 - \Delta \theta_2 \]

\[ p = p_1 \cdot p_2 = (1 - \Delta \theta_1) \cdot (1 - \Delta \theta_2) = 1 - \Delta \theta_1 - \Delta \theta_2 + \Delta \theta_1 \cdot \Delta \theta_2 \]

\[ \Delta p = 1 - p = \Delta \theta_1 + \Delta \theta_2 - \Delta \theta_1 \cdot \Delta \theta_2 \]

\[ \Delta \theta' = \lambda \Delta \theta_1 + \lambda \Delta \theta_2 = \lambda (\Delta \theta_1 + \Delta \theta_2) \]

\[ p_1 = 1 - \Delta \theta_1, \quad p_2 = 1 - \Delta \theta_2 \]

\[ p = 1 - \Delta \theta_1 + 1 - \Delta \theta_2 - (1 - \Delta \theta_1)(1 - \Delta \theta_2) = \Delta \theta_1 \cdot \Delta \theta_2 \]

\[ \Delta \theta' = (\lambda \Delta \theta_1)(\lambda \Delta \theta_2) = \lambda^2 (\Delta \theta_1 \cdot \Delta \theta_2) \]
Analytical Expressions

\[ x_1 = (0.5, 0.6), \quad V_1 = 1.0, \quad \theta_1 = 0.9 \]
\[ x_2 = (0.1, 1.3), \quad V_2 = 1.3, \quad \theta_2 = 0.8 \]
\[ x_3 = (1.3, 1), \quad V_3 = 1.6, \quad \theta_3 = 0.95 \]

\[-0.5^2 + 0.6^2 = 0.25 + 0.36 = 0.61\]
\[= 0.25 + 0.36 = 0.61\]
\[= 1.3^2 + 1^2 = 1.69\]
\[= 1.3^2 + 1^2 = 2.69\]

\[\text{Mean:}\]
\[0.9 \cdot 1.0 + 0.08 \cdot 1.3 + 0.02 \cdot 1.6 =\]
\[0.9 + 0.10 + 0.03 = 1.03\]

\[\text{Variance:} \quad (0, 0)\]

\[
\begin{array}{ccc}
0.9 & -0.3 & 0.001 = 0.999 \\
0.08 & 0.27 & 0.08 = 0.006 \\
0.02 & 0.57 & 0.06 = 0.007 \\
\end{array}
\]

\[\sqrt{0.014} \approx 0.12\]
\[ V = \frac{x_1^2}{2} + \frac{x_2^2}{2} - \left( \frac{x_1 + x_2}{2} \right)^2 \]

\[ \frac{df}{dx_1} = \frac{x_1 - x_2}{2} \]

\[ \frac{df}{dx_2} = \frac{x_2 - x_1}{2} \]

- \( x_1 = 1.0 \) \( \Delta_1 = 0.1 \)
- \( x_2 = 2.0 \) \( \Delta_2 = 0.1 \)

\[ \Delta = \left| \frac{df}{dx_1} \right| \Delta_1 + \left| \frac{df}{dx_2} \right| \Delta_2 \]

\[ \Delta = 0.5 \times 0.1 + 0.5 \times 0.1 = 0.1 \]

Nominal value

\[ \tilde{y} = \frac{1}{2} + \frac{y}{2} - 1.5^2 = 0.5 + 2 - 2.25 = 0.25 \]

Numerical Differentiation

\[ \Delta = \left| f \left( \bar{x}_1 + \Delta_1, \bar{x}_2 \right) - \tilde{y} \right| + \left| f \left( \bar{x}_1, \bar{x}_2 + \Delta_2 \right) - \tilde{y} \right| \]

\[ \Delta = \left| \frac{1.1^2}{2} + 2 - 1.5^2 \right| + \left| \frac{1.21}{2} + 2 - 2.4025 \right| \]

\[ 0.605 + 2 - 2.4025 = 0.2025 \]

- Why we need Monte Carlo?
  - To decrease the computations

- Why we take the tangent?
  - [0, 1] rectangular transformation to have big H's.
\[ y = 0.5 \]

\[ \tan(\pi(r - 0.5)) \]

\[ \Delta \text{ Component with statistical method vs Correlations.} \]

\[ \begin{align*}
\text{Case 1:} & \quad \Delta_1 - \Delta_2 \\
\text{Case 2:} & \quad \Delta_1 + \Delta_2 \\
\end{align*} \]

\[ \Delta = n \cdot \Delta_1 \]

- M.C. ⇒ Statistical Case
- N.Diff ⇒ Interval, you don't need \( \sqrt{\cdot} \)
\[ \frac{1}{\sqrt{1000}} \approx \frac{1}{30} \approx 3\% \]

\[ \Delta \leq 3\sigma \]

\[ \text{std dev} = \sqrt{\frac{\chi^2}{h^2}} \]

\[ V = 0.5 \]

\[ E \left( \frac{\Delta x_1 + \ldots + \Delta x_n}{n} \right)^2 = \frac{\sum \Delta x_i^2}{n} = \frac{V_n}{n} \]

*What problem we are solving: Find the possible range of the indirect measurements.*