Name: __

General comments:

- you are allowed up to 5 pages of handwritten notes;
- if you need extra pages, place your name on each extra page;
- the main goal of most questions is to show that you know the corresponding algorithms; so, if you are running of time, just follow the few first steps of the corresponding algorithm;

Good luck!

1. Use Lagrange multiplier method to solve the following constraint optimization problem: find the point of the line $2x_1 - x_2 = 1$ which is the closest to 0, i.e., in precise terms, minimize the sum $x_1^2 + x_2^2$ under the above constraint.

$$x_1^2 + x_2^2 \rightarrow \min \quad \text{subject to} \quad 2x_1 - x_2 = 1$$

Let

$$L = x_1^2 + x_2^2 + \lambda (2x_1 - x_2 - 1) \rightarrow \min _{x_1, x_2}$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda = 0 \rightarrow x_1 = -\frac{2\lambda}{2} = -\lambda$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda = 0 \rightarrow x_2 = \frac{\lambda}{2}$$

From: $2x_1 - x_2 = 1$

$$2(-\lambda) - \frac{\lambda}{2} = 1$$

$$-4\lambda - \lambda = 2$$

$$-5\lambda = 2$$

$$\lambda = -\frac{2}{5}$$

$$x_2 = -\frac{2}{5} \rightarrow x_2 = \frac{-2}{5} = -\frac{2}{5}$$

$$x_1 = \frac{2}{5}$$

$$2\left(\frac{2}{5}\right) + \frac{1}{5} = \frac{4}{5} + \frac{1}{5} = 1$$

file:///Q:/cs5354.18/test3.html 11/8/2018
2. Suppose that we have two investments, one with expected return 10 and variance 20, another with expected return 20 and variance 10, and we want to have a return of 13. Assuming that these two investments are independent, use the general formulas that we had in class to find the optimal portfolio.

3. Same as in Problem 2, but this time, the two investments are not independent: the covariance is -0.5. Describe the optimal portfolio for this case.

2) Let $\mu_1 = 10$, $\sigma_1^2 = 20$ and $\mu_2 = 13$,

$\mu_2 = 20$, $\sigma_2^2 = 10$

$\Sigma_0 = \Sigma_1 / \sigma_1^2 = \frac{1}{20} + \frac{1}{10} = 0.15$

$\Sigma_1 = \Sigma \mu_i^2 / \sigma_i^2 = \frac{10}{20} + \frac{20}{10} = 2.5$

$\Sigma_2 = \Sigma \mu_i^2 / \sigma_i^2 = \frac{10^2}{20} + \frac{20^2}{10} = 45$

$q = \frac{\Sigma_1 - \mu_2 \Sigma_0}{\Sigma_1 - \Sigma_0 \Sigma_2} = \frac{2.5 - (13 \times 0.15)}{(2.5)^2 - (0.15 \times 45)} = -1.1$

$b = \frac{(1 - q \Sigma_1)}{\Sigma_0} = \frac{(1 + [1.1 \times 2.5])}{0.15} = 25$

$\omega_1 = (-1)(\frac{10}{20}) + (25)(\frac{1}{20}) = 0.7$ to invest I

$\omega_2 = 1 - \omega_1 = 1 - 0.7 = 0.3$ to invest II
Given: \( u_1 = 10 \), \( \sigma_1^2 = 20 \)

\( u_2 = 20 \), \( \sigma_2^2 = 10 \)

\[ c_{12} = -0.5 \]

Since we have two investments, the weights are not affected by the covariance, hence:

From: \( \sum w_i = 1 \) \( w_1 + w_2 = 1 \) \( (1) \)

From: \( \sum w_i u_i = \mu_0 \) \( 10w_1 + 20w_2 = 13 \) \( (2) \)

Equ. (1) \( \times 10 \) \( 10w_1 + 10w_2 = 10 \) \( (3) \)

Equ. (2) - (3) \( 0 + 10w_2 = 3 \)

\[ w_2 = \frac{3}{10} = 0.3 \]

\[ w_1 = 1 - w_2 = 0.7 \]
4.7. Assume that we have ten estimates for the company's worth: three estimates of 2 Billion dollars, five estimates of 3 Billions, and two outliers: an over-pessimistic estimate of 0 Billion, and an over-optimistic estimate of 10 Billions.

4. What will be the combined estimate if we use the standard least squares methods (i.e., $l^2$).

5. What will be the combined estimate if we use the $l^1$ method? Explain in what sense this method is more robust.

6. What is the general class of robust techniques that includes both $l^2$ and $l^1$ as particular cases? What is the justification for using methods from this class?

7. Describe the first few steps of an algorithm for providing the $l^p$-estimate for $p = 1.5$. (No need to actually perform the computations).

\[ \text{estimates are } 2, 2, 2, 3, 3, 3, 3, 3, 3, 0, 10 \]

(4) Combined estimate by $l^2 = \frac{3(2) + 5(3) + 0 + 10}{10} = 3.1$

i.e. Arithmetic average.

(5) Combined estimate by $l^1$ is the median given by

\[ 0 < 2 < 2 < 2 < 3 < 3 < 3 < 3 < 10 \]

\[ \text{median } = \frac{3 + 3}{2} = 3 \]

(b) $l^1$ is more robust because the estimate is not affected by the outliers 0, 10

(6) General class that includes $l^1$ and $l^2$ is $l^p$ method.

(b) Justification is that the estimate $L(e)$ is scale invariant with respect to the error $e$.

\[ L(e) = \text{cost} \, |e|^p \]
The $L^p$ Algorithm with $p = 1.5$

Idea:  
\[ \min_{\beta} \sum_{k=1}^{n} \left| y_{k} - f(x_k, \beta) \right|^{1.5} = \min_{\beta} \sum_{k=1}^{n} \frac{\left( y_{k} - f(x_k, \beta) \right)}{\left| y_{k} - f(x_k, \beta) \right|^{0.5}} \]

where $\beta$ is the parameter to be estimated.

Given $x_k, y_k$ for $k = 1, \ldots, n$, we want to find $\beta$ for which

\[ \sum_{k=1}^{n} \left| y_{k} - f(x_k, \beta) \right|^{1.5} \to \min. \]

Step 1: We use the least squares method to find $\beta^{(0)}$ for which

\[ \sum_{k=1}^{n} \left| y_{k} - f(x_k, \beta) \right|^{2} \to \min. \]

Step $s, s = 2, 3, \ldots $ we take $\beta^{(s-1)}$ and compute

\[ |e_{k}^{(s)}| = \left| y_{k} - f(x_k, \beta^{(s-1)}) \right| \]

for which

\[ \sum_{k=1}^{n} \frac{\left( y_{k} - f(x_k, \beta) \right)^{2}}{|e_{k}^{(s)}|^{0.5}} \to \min. \]

This is $\beta^{(s)}$.

Next, we continue with steps until $|\beta^{(s)} - \beta^{(s-1)}| \leq \varepsilon$, i.e., the required accuracy is obtained.
8-10. An investor placed her money into two hedge funds. The first one led to annual returns of 10%, 5%, 5%, 5%, and 10%. The second one lead to annual returns of 9%, 0%, 9%, 9%, and 9%.

8. Which of the two investments leads to better end results? \[10/10\]

9. Which of the two investments will the investor prefer if he/she follows the peak-end rule? \[10/10\]

10. What is the justification for the peak-end rule? \[10/10\]

\[\begin{align*}
\text{Invest I total return} &= 1.1 \times 1.05 \times 1.05 \times 1.05 \times 1.1 = 1.4607 \\
\text{Invest II total return} &= 1.09 \times 1.09 \times 1.09 \times 1.09 = 1.416 \\
\text{In reality, invest II is better.}
\end{align*}\]

\[\begin{array}{c|c|c}
\text{Peak-end rule} & \text{Invest I} & \text{Invest II} \\
\hline
\text{First} & 10 & 9 \\
\text{Last} & 10 & 9 \\
\text{Minimum} & 5 & 0 \\
\text{Maximum} & 10 & 9 \\
\end{array}\]

Comparing I and II, by peak end rule II is better.

\[\begin{align*}
\text{Justification of peak-end rule is that, the combined satisfaction or experience is:} \\
\text{(1) Associative} & \quad \text{(3) Scale invariant} \\
\text{(2) Idempotent} & \quad \text{(4) Shift invariant}
\end{align*}\]