Solution to Problem 11

Task.

- What can you say about the Kolmogorov complexity of a string 0101...01 (repeated 1000 times). Is it C-random for C = 10? Explain your answer.

- Prove that Kolmogorov complexity is not computable.

What can we say about the Kolmogorov complexity of the given string? By definition, the Kolmogorov complexity $K(x)$ of a string $x$ is the length $\text{len}(p_0)$ of the shortest program $p_0$ that prints this string.

Kolmogorov complexity is not computable, so we cannot compute $K(x)$, but we can find an upper bound for $K(x)$. Indeed, if we have a program $p$ that prints the string $x$, then, by definition, $K(x) \leq \text{len}(p)$.

We can write a simple loop to print the given sequence:

```java
for(int i=1, i<=1000, i++)
    System.out.print("01");
```

This program $p$ has $\text{len}(p) = 48$ symbols, so we conclude that $K(x) \leq 48$.

Is this sequence random? Random means $K(x) \geq \text{len}(x) - C$, for $C = 10$. In our case,

$$\text{len}(x) = 2 \cdot 1000 = 2000,$$

so

$$\text{len}(x) - C = 2000 - 10 = 1990,$$

while $K(x) \leq 48$. Clearly, $K(x) < \text{len}(x) - C$, so the sequence $x$ is not random.

Proof. Let us prove this statement by contradiction. Let us assume that that the Kolmogorov complexity is computable, i.e., that there exists an algorithm that, given a string $x$, computes $K(x)$. Then, given a string $x$, we can check check whether this string is $C$-random, i.e., whether $K(x) \geq \text{len}(x) - C$, as follows:

- first, we use the algorithm for computing $K(x)$ – whose existence we assumed – to compute $K(x)$;

- then, we check whether this computed value is greater than or equal to $\text{len}(x) - C$. 


Thus, we get an algorithm that checks whether a string is \( C \)-random. But in the lecture, we proved that such algorithm is not possible. This contradiction proves that our assumption cannot be true. Thus, no algorithm is possible that computes \( K(x) \).