The Laws of Excluded Middle and Contradiction in Checklist Paradigm Based Fuzzy Interval Logic

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Abstract—This paper continues a study in fuzzy interval logic based on the Checklist Paradigm (CP) semantics of Bandler and Kohout. The law of excluded middle and the law of contradiction are investigated in the fuzzy interval logic system of negation, \([\neg_{Bot}, \neg_{Top}]\), which was defined by the Nicod (NOR) and the Sheffer (NAND) connectives of \(m_1\) interval system, respectively.

Both laws don’t hold in the fuzzy logic with a classic negation, \(\neg a = 1 - a\); however, they do hold both with \(\neg_{Bot}\) and with \(\neg_{Top}\) in the checklist paradigm based fuzzy interval logic.


I. INTRODUCTION

This study continues fuzzy interval logic based on the Checklist Paradigm semantics of Bandler and Kohout. The fuzzy logic default into 2-valued crisp logic for the values 0 and 1, however, the logical operations with the values in the open interval \((0, 1)\), other than 0 and 1, generate different values depending on the definition of logical connective which was applied. The question how connectives of the same type differ between the boundary points, i.e. within the open interval \((0, 1)\), has been well researched. For instance, the fuzzy logical AND (\(\land\)), i.e. a t-norm operator, such as \textit{minimum} (\(\land_m\)), algebraic product (\(\land_a\)), or drastic product (\(\land_d\)) yields a hierarchy of their values as follows:

\[ a \land_d b \leq a \land a b \leq a \land_m b \]

where \(a \land b = \max(0, a + b - 1)\), \(a \land_a b = ab\), \(a \land_m b = \min(a, b)\), and \(a \land_d b = a\) if \(b = 1\); \(b\) if \(a = 1\); 0 otherwise. Similarly, the fuzzy OR (\(\lor_a\)) yields an hierarchy

\[ a \lor_m b \leq a \lor_a b \leq a \lor b \leq a \lor_d b \]

where \(a \lor_m b = \max(a, b)\) (maximum), \(a \lor_a b = a + b - ab\) (algebraic sum), \(a \lor_d b = \min(1, a + b)\) (bounded sum) and \(a\lor_d = a\) if \(b = 0\); \(b\) if \(a = 0\); 1 otherwise. On the other hand, the implication connectives such as \(\text{nlu\text{"a}kiewicz}(\rightarrow)\), Kleene-Dienes (\(\rightarrow DK\)), Reichenbach (\(\rightarrow KD\)), Goguen (\(\rightarrow G\)), Gödel (\(\rightarrow S\)) yield an inequality relationship as follows:

\[ a \rightarrow S b \leq a \rightarrow G b \leq a \rightarrow L b : \text{R-implications,} \]

\[ a \rightarrow W b \leq a \rightarrow E b \leq a \rightarrow K b \leq a \rightarrow L b : \]

S-QL implications, where \(a \rightarrow G b = \min(1, b/a)\); \(a \rightarrow L b = \min(1, 1 - a + b)\); \(a \rightarrow K b = \min(1, 1 - a + ab)\); \(a \rightarrow E b = \max(1 - a, b)\); \(a \rightarrow Z b = \max(1 - a, \min(a, b))\); \(a \rightarrow W b = \min(\max(1 - a, b), \max(a, 1 - a, 1 - a + ab))\) [1],[2]. However, inter-relations of interval systems of connectives are known much less.

Bandler and Kohout derived five interval systems of fuzzy logic, \(m_1\), \(m_2\), \ldots, \(m_5\), based on the Checklist paradigm in 1979 [2]. Since then, the logic systems of connectives that can be generated from the interval of implication by group transformations have been investigated systematically; in particular, the \(m_1\) interval logic system of 16 connectives has been investigated in depth [3]-[7]. In their \(m_1\) logical system, ten 2-argument connectives such as \(\land, \lor, \rightarrow, \ldots, \equiv (\text{IFF})\) and \(\oplus (\text{XOR})\) yield the interval pairs of connectives \((\text{conbot} \leq m_1 \leq \text{contop})\) where its implication \((\rightarrow)\) yields Łukasiewicz and Kleene-Dienes implication for its \(\text{TOP-BOT}\) pair of interval, in particular. However, a unary connective such as a negation \((\neg a)\) and an identity \((a)\) did not yield interval but just singleton: i.e. \(\neg a = 1 - a\) and \(a\). From this question, Kim and Kohout proposed the alternative negations in [8] defined by the Sheffer (NAND), the Nicod (NOR) and a pair of \(<\rightarrow, 0 >\), which fuzzifies \(\neg a\), generating the intervals like the fuzzified 2-ary connectives. Since the \(\text{TOP-BOT}\) pair of these generated negation interval lacked of an involutive property on the surface, some pseudo properties such as a nearly involutive property and a convergence of iteration of each \(\text{TOP-BOT}\) pair were also investigated in [9].

When the fundamental laws in the crisp logic are extended to a fuzzy logic, most of laws such as De Morgan’s law and associativity, etc., do hold in the fuzzy logic; however, both the law of excluded middle and the law of contradiction fail with a classic negation in singleton, \(\neg a = 1 - a\) in the fuzzy logic. Thus, it raises a question whether both laws hold with \(\text{TOP-BOT}\) pair of the fuzzy interval negation and with those pair of fuzzy interval identity or not.

In this paper, an interval system of fuzzy identity \((a)\) is defined using the law of idempotence. Then, both the law of excluded middle and the law of contradiction are examined...
with the interval of fuzzy negation ($\neg a$) and that of $a$ in $m_1$
logic system based on the Checklist paradigm.

II. INTERVAL LOGIC SYSTEM OF CONNECTIVES
GENERATED BY THE CHECKLIST PARADIGM

Many valued logic interval-based reasoning plays an important role in fuzzy logic and other many valued extensions of crisp logic. To be of use in a diversity of application domains, the interval-valued inference systems require formal semantics.

The formal semantics that is derived by means of a mathematical method, and which also has a sound ontological and epistemological base is provided by the so called checklist paradigm developed by Bandler and Kohout [2],[3],[4],[5]. The checklist paradigm has given interesting theoretical results, shedding light not only on the semantics of various many valued logic connectives, but also on the true methodological importance of fuzzy methods in approximate reasoning based on the interval methods.

In its most general form, the checklist paradigm pairs the distinct connectives of the same logical type to provide the bounds for interval-valued approximate inference. In the Checklist Paradigm by Bandler and Kohout
A checklist template $Q$ is a finite family of properties $(P_1, P_2, ..., P_n)$; With a template $Q$, and a given proposition $A$, one can associate a specific checklist $Q_A = \langle Q, A \rangle$. A valuation $f_A$ of a checklist $Q_A$ is a function from $Q$ to $\{0, 1\}$.

The value $a_Q$ of the proposition $A$ with respect to a template $Q$ (which is the summarized value of the valuation $f_A$) is given by the formula

$$a_Q = \frac{1}{n} \sum_{i=1}^{n} p_i^A$$

where $n = |Q|$ and $p_i^A = f_A(P_i)$.

A fine valuation structure of a pair of propositions $A$, $B$ with respect to the template $Q$ is a function $f_{A,B}^Q$ from $Q$ into $\{0, 1\}$ assigning to each attribute $P_i$ the ordered pair of its values $(p_i^A, p_i^B)$.

Let $\alpha_{j,k}$ be the cardinality of the set of all attributes $P_i$ such that $f_{A,B}^Q(P_i) = \langle j, k \rangle$. Then, there are the following constraint on the values: $\alpha_{00} + \alpha_{01} + \alpha_{10} + \alpha_{11} = n$. Further, $r_0 = \alpha_{00} + \alpha_{01}$, $r_1 = \alpha_{10} + \alpha_{11}$, $c_0 = \alpha_{00} + \alpha_{10}$, $c_1 = \alpha_{01} + \alpha_{11}$ are defined.

These entities can be displayed systematically in a constraint table. In such a table, the inner fine-summarization structure consists of the four $\alpha_{j,k}$ appropriately arranged, and of margins $c_0, c_1, r_0, r_1$ (see Fig. 1).

Now let $F$ be any logical propositional function of propositions $A$ and $B$. For $i, j \in \{0, 1\}$, let $f(i, j)$ be the classical truth value of $F$ for the pair $i, j$ of truth values; let $u(i, j) = \alpha_{i,j}/n$, the ratio of the number in the $ij$-cell of the constraint table, to the grand total. Then what we define the (non-truth-functional) fuzzy assessment of the truth of the proposition $F(A, B)$ to be

$$m(F(A, B)) = \sum_{i,j} f(i, j) \cdot u_{ij}.$$
distinct measures below:

- \(m_2(F) = \frac{a_{11}}{a_{00} + a_{11}} = 1 - \frac{a_{10}}{a_1} \): performance measure.
- \(m_3(F) = u_{11} \lor (u_{00} + u_{a1})\): by performance and by default.
- \(m_4(F) = m_3 \land \neg m_3\): lower contrapositivization of \(m_3\).
- \(m_5(F) = m_2 \lor \frac{a_2}{a} = m_2 \lor (u_{00} + u_{a1})\): the less conservative performance.

In each of four measures, the interval of logical connective of implication (\(\to\)) is generated and further builds five distinct interval systems of connectives \(m_1 - m_5\) in the next section.

### B. Five Implication Operator Based Interval Systems of Bandler and Kohout

The structure of five fuzzy interval systems \(m_1 - m_5\), based on the Checklist paradigm by Bandler and Kohout in [2] is generated by a distinct measure that performs the summarization of the information contained in certain well-defined binary structures called *fine structures* as it is described in the above section. The interval produced by a measure \(m_i\) pair of connectives of one type can be generically characterized by the following inequality:

\[
\text{conbot} \leq m_i \leq \text{contop}, \quad i \in \{1, 2, 3, 5\}
\]

For the more details of the checklist paradigm and its various uses, refer to the papers [2] -[7],[12].

The following five inequalities linking the interval bounds for implication operators \([\neg_{\text{bot}}, \neg_{\text{top}}]\) with corresponding measures, \(m_i, \quad i \in \{1, 2, 3, 4, 5\}\) have been generated from five different measures described in Sec.II-A [2].

1. The Kleene-Dienes implication (KD) and Łukasiewicz implication (L) respectively, are attainable lower and upper bounds of \(m_1\):
   \[
   \max(1-a, b) \leq m_1(\to) \leq \min(1, 1-a+b).
   \]

2. A certain new function of \((a, b)\) and the Goguen-Gaines (G43) implication (the left-hand side) are respectively attainable lower and upper bounds of \(m_2\):
   \[
   \max(0, (a+b-1)/a) \leq m_2(\to) \leq \min(1, b/a).
   \]

3. Another function of \((a, b)\) and the Early Zadeh implication (EZ) are respectively attainable lower and upper bounds of \(m_3\):
   \[
   \max(a+b-1, 1-a) \leq m_3(\to) \leq \max[\min(a, b), 1-a].
   \]

4. Still another function of \((a, b)\) and the Wilmott implication (W) respectively, are attainable lower and upper bounds of \(m_4\):
   \[
   \min[\max(a+b-1, 1-a), \max(b, 1-a-b)] \leq m_4(\to) \leq \min[\max(1-a, b), (\max(a, 1-b), \min(b, 1-a))].
   \]

5. Yet another function of \((a, b)\) and one of G43 respectively, are attainable lower and upper bounds of \(m_5\):
   \[
   \max[(a + b - 1)/a, 1-a] \leq m_5(\to) \leq \max[\min(1, b/a), 1-a].
   \]

In \(m_1\) system, a Kleene-Dienes logic system forms a a lower bound of the interval \((m_1(\neg_{\text{bot}}))\) while a Łukasiewicz logic system forms a upper bound of the interval \((m_1(\neg_{\text{top}}))\).

From this implicational interval pair \([m_1(\neg_{\text{bot}}), m_1(\neg_{\text{top}})]\), all 16 pairs of connectives of corresponding interval systems could be generated by group transformation of logic. Among 16 pairs of logical connectives, 10 of them are genuine interval pairs: \([\neg, \to, \vee, \land, \neg\neg, \vee, \land, \equiv (IFF), \oplus (XOR)]\). The rest of 6 connectives, however, collapse into a single point, not an interval: \([\text{true, false, identity (a, b), negation (a, b)}]\); see [5] for the details of intervals \([\text{conbot, conbot}]\) of 16 connectives. It has been investigated systematically over the years in the series of papers [6]-[7],[13]. These 16 connectives of measure \(m_1 - m_5\) could be generated by group transformation. 8 of these interval pairs of connectives of \(m_1\) system are shown at Table 1 in section III. In the following section, the operations of group transformation is described.

### III. CHARACTERIZATION OF LOGICS BY GROUP TRANSFORMATION

The \(m_1\) interval system of logic can be generated by group transformations of logical connectives.

**Definition 1:** Group Transformations in Logic:

Let \(f\) be one of the 10 two-argument propositional connectives of a logic system \(\{\to, \neg, \ldots, \equiv, \oplus\}\) and \(\neg\) be an involutive negation. Then, the following transformations over \(f\) are defined:

1. \(I(f) = f(x, y)\) : Identity transformation
2. \(D(f) = \neg f(\neg x, \neg y)\) : Dual transformation
3. \(C(f) = f(\neg x, \neg y)\) : Contradual transformation
4. \(N(f) = \neg f(x, y)\) : Negation transformation

It is well known that for the crisp (2-valued) logic these transformations determine the Piaget group which is a realization of the abstract Klein 4-element group.

The new non-symmetrical transformations below were added to the above 4 basic transformations by Bandler and Kohout in 1979 [14],[15]. These non-symmetrical transformations enrich the algebraic structure of logical transformations.

**Definition 2:** 8-element Group Transformations: Bandler-Kohout

- \(L.C(f) = f(\neg x, y)\) : Left Contradual
- \(R.C(f) = f(x, \neg y)\) : Right Contradual
- \(L.D(f) = \neg f(\neg x, y)\) : Left Dual
- \(R.D(f) = \neg f(x, \neg y)\) : Right Dual

This enlarged set of transformations \(T = \{I, D, C, N, L.C, R.C, L.D, R.D\}\) forms 8-element group \(T, \circ\) called \(S_{2\times2\times2}\) group. The equation, \(N^2 = C^2 = D^2 = L.C^2 = L.D^2 = R.C^2 = R.D^2 = I\), provides sufficient information to identify this group. Its group operations are shown in [17] in detail. When \(T = \{I, D, C, N, L.C, R.C, L.D, R.D\}\) are applied to Łukasiewicz implication \((\neg_{\text{top}})\) and to Kleene-Dienes implication \((\neg_{\text{bot}})\) of \(m_1\) system of logic, respectively, they yield the closed set of connectives in Table I [6],[17]. This result of \(m_1\) interval system of 8 connectives is further summarized in Table II. As Table II shows, neither negation nor identity belonged to the interval system \(m_1\).

The graph of transformations showing a more detailed structure of the transformation of connectives that realize
Table I

Table I: Transformation of m₁ system: <→_{top}, ¬_{Bot}>

<table>
<thead>
<tr>
<th>Transformation of Connective</th>
<th>Type of Interval Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_{11} = I(→_{Top}) = \max(0, a + b - 1) )</td>
<td>↓_{Top}</td>
</tr>
<tr>
<td>( g_{12} = C(→_{Top}) = \min(1, 0 + b) )</td>
<td>↑_{Top}</td>
</tr>
<tr>
<td>( g_{13} = N(→_{Top}) = \max(0, a + b) )</td>
<td>¬_Bot</td>
</tr>
<tr>
<td>( g_{14} = L(→_{Bot}) = \min(1, 0 + b) )</td>
<td>↑_{Top}</td>
</tr>
<tr>
<td>( g_{15} = LD(→_{Bot}) = \max(0, 1 + b) )</td>
<td>( \nabla_{Top} )</td>
</tr>
<tr>
<td>( g_{16} = RC(→_{Bot}) = \min(1, 2 - a - b) )</td>
<td>↑_{Top}</td>
</tr>
<tr>
<td>( g_{17} = RD(→_{Bot}) = \max(0, a + b - 1) )</td>
<td>( \land_{Bot} )</td>
</tr>
</tbody>
</table>

The transformation \( g_{1b} = I(→_{Bot}) = \max(1 - a, b) \) is defined by \( \rightarrow_{Bot} \).

The transformation \( g_{2b} = C(→_{Bot}) = \max(a, 1 - b) \) is defined by \( \leftarrow_{Bot} \).

The transformation \( g_{3b} = D(→_{Bot}) = \min(1 - a, b) \) is defined by \( \downarrow_{Bot} \).

The transformation \( g_{4b} = N(→_{Bot}) = \min(a, 1 - b) \) is defined by \( \Rightarrow_{Bot} \).

The transformation \( g_{5b} = LD(→_{Bot}) = \max(a, b) \) is defined by \( \nabla_{Bot} \).

The transformation \( g_{6b} = LD(→_{Bot}) = \min(1, 1, a - b) \) is defined by \( \top_{Bot} \).

The transformation \( g_{7b} = RC(→_{Bot}) = \max(1, 1, a - b) \) is defined by \( \bot_{Bot} \).

The transformation \( g_{8b} = RD(→_{Bot}) = \min(a, b) \) is defined by \( \land_{Bot} \).

Table II

Table II: Top/Bot Connectives in m₁ System

<table>
<thead>
<tr>
<th>No.</th>
<th>Logical Type</th>
<th>Valuation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Conjunction (AND) ( a&amp;b )</td>
<td>( \max(0, a + b - 1) )</td>
</tr>
<tr>
<td>2</td>
<td>Non-Conjunction (Nicod) ( a \downarrow_{b} )</td>
<td>( \min(1, 0 + b) )</td>
</tr>
<tr>
<td>3</td>
<td>Non-Inversion (Sheffer) ( a \mid_{b} )</td>
<td>( \max(1, 1 - a + b) )</td>
</tr>
<tr>
<td>4</td>
<td>Disjunction (OR) ( a \vee_{b} )</td>
<td>( \min(1, a + b) )</td>
</tr>
<tr>
<td>5</td>
<td>Non-Inverse Implication ( a \leftarrow_{b} )</td>
<td>( \max(0, 1 - a) )</td>
</tr>
</tbody>
</table>
| 6   | Implication \( a 
{\to}_{b} \) | \( \min(1, a + b) \) |
| 7   | Equivalence (IFF) \( a \equiv_{b} \) | \( \max(0, a - b) \) |
| 8   | Exclusion (XOR) \( a \oplus_{b} \) | \( \min(2 - a, a + b) \) |

Thus, \( \neg a \) defined by Sheffer generates the interval of negation

\[ [\bot_{Bot}, \top_{Bot}] = [1 - a, \min(1, 2(1 - a))] \]

Similarly, the negation by Nicod \( \neg_{N} \) could be defined by applying \( \downarrow_{Top} \) and \( \downarrow_{Bot} \) which are duals of \( \top_{Bot} \) and \( \top_{Top} \), respectively.

\[ \neg_{N} a = \begin{cases} a \downarrow_{Top} a = 1 - a & \text{if } a \top_{Top} a, \\ a \downarrow_{Bot} a = \max(0, 1 - 2a) & \text{if } a \top_{Bot} a. \end{cases} \]

Hence, the interval of \( \neg_{N} a \) generated by Nicod is:

\[ [\downarrow_{Bot}, \downarrow_{Top}] = [\max(0, 1 - 2a), 1 - a]. \]

IV. INTERVAL NEGATIONS IN MANY-VALUED LOGICS

As Table II shows, 8 connectives could be intervalized through a group transformation; however, the rest of 6 unary connectives were not yet: \{true, false, identity \( a, b \), negation \( \neg a, \neg b \).\} In [8], Kim and Kohout defined the interval of negation in m₁ logical system using the interval of Sheffer (\( \lnot \)) and Nicod (\( \downarrow \)) connectives.

First, the negation by Sheffer \( \neg_{S} \) is defined by employing \( \downarrow_{Top} \) and \( \downarrow_{Bot} \), respectively.

\[ \neg_{S} a = \begin{cases} a \downarrow_{Top} a = \min(1, 2(1 - a)) & \text{if } a \top_{Top} a, \\ a \downarrow_{Bot} a = 1 - a & \text{if } a \top_{Bot} a. \end{cases} \]

Although both \( \neg_{Top} \) and \( \neg_{Bot} \) are a non-involutive TOP-BOT pair of fuzzy interval negations, we have examined the convergence of their iteration and their nearly involutive property in [9].

V. INTERVAL OF a BY IDEMPOTENCE

The unary connective identity of \( a \) can be similarly intervalized to an interval \([a_{Bot}, a_{Top}]\) by extending the law of idempotence in the crisp logic: \( a \land a = a \) and \( a \lor a = a \).
A. Intervalization of \( a \) on \( m_1 \) Defined by the AND Connective

Since the AND connective (\( \land \)) of the interval system \( m_1 \) yields a \( \top - \bot \) pair by means of right dual (RD) transformation of \( \langle \top \rightarrow \bot, \top \rangle \) as in Table I,

\[
a \land b = \begin{cases} 
\land_{\top} b = \min(a, b) \\
\land_{\bot} b = \max(0, a + b - 1), 
\end{cases}
\]

the identity of \( a \) generated by the logical AND, \( a_{A} \), is yielded in two forms:

\[
a_{A} = \begin{cases} 
\land_{\top} a = a \\
\land_{\bot} a = \max(0, 2a - 1), 
\end{cases}
\]

Thus, the interval of \( a_{A} \) is

\[
[a_{A_{\top}}, a_{A_{\bot}}] = [\max(0, 2a - 1), a].
\]

B. Intervalization of \( a \) in \( m_1 \) Defined by the OR Connective

Since the OR connective (\( \lor \)) of the interval system \( m_1 \) similarly yields a \( \top - \bot \) pair of connectives by means of left contra dual (LC) transformation of \( \langle \top \rightarrow \bot, \bot \rangle \) as below,

\[
a \lor b = \begin{cases} 
\lor_{\top} b = \min(1, a + b) \\
\lor_{\bot} b = \max(a, 2b), 
\end{cases}
\]

the identity of \( a \) generated by the logical OR, \( a_{O} \), also appears in two forms:

\[
a_{O} = \begin{cases} 
\lor_{\top} a = \min(1, 2a) = D(a \land_{\bot} a) \\
\lor_{\bot} a = a = D(a \land_{\top} a), 
\end{cases}
\]

Thus, the interval of \( a_{O} \) by the logical OR connective is:

\[
[a_{O_{\bot}}, a_{O_{\top}}] = [a, \min(1, 2a)].
\]

C. Interval of \( a \) in \( m_1 \) logic system

From two intervals of identity \( id_{A} \) and \( id_{O} \), an interval of identity (\( [a_{A_{\top}}, a_{A_{\bot}}] \)) may be generated by combining both intervals since \( a \) is both the lower bound of \( a_{O} \) and the upper bound of \( a_{A} \), respectively. Thus, the interval of identity is defined as follows.

Definition 4: Interval of Identity

\[
[a_{A_{\top}}, a_{A_{\bot}}] = [a_{A_{\top}}, a_{A_{\bot}}] \cup [a_{O_{\bot}}, a_{O_{\top}}] = [\max(0, 2a - 1), \min(1, 2a)]
\]

where \( a_{mid} = a \land_{\top} a = a \lor_{\bot} a = a \).

Thus, OR(\( \lor_{\top} \)) and AND(\( \land_{\bot} \)) form the TOP and BOT system of identity, respectively, while \( a \), is a median value of the interval of fuzzy identity, \( a_{mid} \).

In addition:

\[
[\neg a, \neg a] = 1 - [a_{A}, a_{A}].
\]

Both interval of fuzzy negation and the interval of fuzzy identity are depicted in Fig. 3.

The margins of imprecision in the interval of connectives were formulated in Gap Theorem.

\[
\begin{align*}
\text{Theorem 5: Gap Theorem 1. (Bandler and Kohout [5])} \\
a \land_{\top} b - a \land_{\bot} b & = a \lor_{\top} b - a \lor_{\bot} b \\
& = a \land_{\top} b - a \land_{\bot} b \\
& = \min(\varphi a, \varphi b).
\end{align*}
\]

Similarly, the margins of imprecision can be formulated for a \( \top - \bot \) pair of negation and that of identity below.

\[
\begin{align*}
\text{Theorem 6: Gap Theorem 2.} \\
\neg_{\top} a - \neg_{\bot} a & = a_{T} - a_{B} \\
& = \min(2a, 2(1 - a)) \\
& = 2\varphi a.
\end{align*}
\]

Hence, the margins of imprecision can be directly measured by the degree of fuzziness \( \varphi \) where \( \varphi a = \min(a, 1 - a) \).

VI. LAWS OF EXCLUDED MIDDLE AND CONTRADICTION IN FUZZY INTERVAL LOGIC

A. Law of Excluded Middle

This section investigates the law of excluded middle with the interval of negation and that of identity which were defined in Section V where

\[
[a_{A_{\top}}, a_{A_{\bot}}] = [\max(0, 2a - 1), \min(1, 2a)], \quad \text{and}
\]

\[
[\neg_{\top} a, \neg_{\bot} a] = [\max(0, 1 - 2a), \min(1, 2(1 - a))].
\]

Since each of \( a_{T}, a_{B}, \neg a \) and \( \neg a \) was generated using \( \lor_{\top}, \lor_{\bot}, \land_{\top}, \land_{\bot} \) respectively, which belongs to the closed set of \( m_1 \) logic system of Łukasiewicz connective, i.e. \( \langle \top_{\top}, \top_{\bot}, \bot_{\bot} \rangle \), as it is shown in Table I, \( \top_{\top} \) is applied for \( \lor \) to investigate the law of excluded middle, \( a \lor \neg a \) in each case where \( a \lor_{\top} b = \min(a, 1 + b) \).

In addition, since \( \neg a = 1 - a \) and \( \neg a = 1 - a B \), an excluded middle is computed with a pair of \( a_{T}, \neg a \) and that of \( a_{B}, \neg a \) with \( \lor_{\top} \).

\[
\begin{align*}
\text{Theorem 5: Gap Theorem 1. (Bandler and Kohout [5])} \\
a \land_{\top} b - a \land_{\bot} b & = a \lor_{\top} b - a \lor_{\bot} b \\
& = a \land_{\top} b - a \land_{\bot} b \\
& = \min(\varphi a, \varphi b).
\end{align*}
\]

\[
\begin{align*}
\text{Theorem 6: Gap Theorem 2.} \\
\neg_{\top} a - \neg_{\bot} a & = a_{T} - a_{B} \\
& = \min(2a, 2(1 - a)) \\
& = 2\varphi a.
\end{align*}
\]
Thus, the law of excluded middle holds with TOP-BOT pair of negation and identity, vice versa, in the fuzzy interval logic of $m_1$ system of Łukasiewicz connective: $<_{aT}, \neg_{B}, \vee_{Top}>$ and $<_{aB}, \neg_{T}, \vee_{Top}>$.

B. Law of Contradiction

We can investigate the law of contradiction in the intervals of $[\neg_B, \neg_{Top}]$ and $[a_B, a_{Top}]$, similarly. Since each of $a_T$, $a_B$, $\neg a$ and $\neg B$, are generated using $\vee_{Top}$, $\wedge_{Bot}$, $\neg_B$, and $\neg_{Top}$, respectively, which also belongs to the closed set of $m_1$ logic system of $\{\neg_{Top}, \neg_{Bot}, \vee_{Bot}, \wedge_{Bot}, \vee_{Top}, \neg_B, \neg_{Top}\}$ in Table I. $\wedge_{Bot}$ is applied for $\wedge$ to compute the law of contradiction, $a \wedge \neg a$, in each case, where $a \wedge_{Bot} b = \max(0, a + b - 1)$. Since $\neg_B a = 1 - a_T a$ and $\neg_{T} a = 1 - a_B a$, we also compute a degree of contradiction with a pair of $<_{aT}, \neg_{B}>$ and that of $<_{aB}, \neg_{T}>$ with $\wedge_{Bot}$.

$$a_T \wedge_{Bot} (\neg_B a) = \min(1, 2a) \wedge_{Bot} \max(0, 1 - 2a)$$
$$= \max(0, \min(1, a) + \max(0, 1 - 2a) - 1)$$
$$= 0$$ (3)

$$a_B \wedge_{Bot} (\neg_{T} a) = \max(0, 2a - 1) \wedge_{Bot} \min(1, 2(1 - a))$$
$$= \max(0, \max(0, 2a - 1) + \min(1, 2(1 - a) - 1))$$
$$= 0$$ (4)

Thus, the law of contradiction holds with with TOP-BOT pair of negation and identity, vice versa, in the fuzzy interval logic of $m_1$ system: $<_{aT}, \neg_{B}, \wedge_{Bot}>$ and $<_{aB}, \neg_{T}, \vee_{Top}>$.

VII. Conclusions

Both the fuzzy identity (a or b) and the fuzzy negation $\neg a = 1 - a$ which were collapsed into a single point are now extended to their intervals of $[a_B, a_{Top}]$ and $[\neg_{Bot}, \neg_{Top}]$ in the $m_1$ logic system of intervals based on the semantics of Checklist paradigm by Bandler and Kohout. With both interval of fuzzy negation and that of fuzzy identity, the law of excluded middle and the law of contradiction were investigated at TOP-BOT pair of both intervals. Both laws fail with the classic negation and an identity of a single point, i.e. $\neg a = 1 - a$, which is also a median value of the interval of negation $[\neg_{Bot}, \neg_{Top}]$, and a median of $[a_{Bot}, a_{Top}]$, respectively. However, they do hold with at TOP-BOT pair of both intervals $[a_B, a_{Top}]$ and $[\neg_{Bot}, \neg_{Top}]$ when they were computed with the above interval and a corresponding OR_{Top} (i.e. $\vee_{T}$) and AND_{Bot} (i.e. $\wedge_{B}$).

REFERENCES