SET-VALUED EXTENSIONS OF FUZZY LOGIC: CLASSIFICATION THEOREMS

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Abstract

Experts are often not 100% confident in their statements. One of the most widely used approaches to describe the different degrees of confidence is the approach of *fuzzy logic*. In traditional fuzzy logic, the expert's degree of confidence in each of his or her statements is described by a number from the interval $[0, 1]$. However, due to similar uncertainty, an expert often cannot describe his or her degree by a *single* number. It is therefore reasonable to describe this degree by, e.g., a *set* of numbers. In this thesis, we show that under reasonable conditions, the class of such sets coincides:

- either with the class of all 1-point sets (corresponding to the traditional fuzzy logic),
- or with the class of all subintervals of the interval $[0, 1]$ (corresponding to the *interval-valued* fuzzy logic),
- or with the class of *all* closed subsets of the interval $[0, 1]$.

Thus, if we want to go beyond the traditional fuzzy logic and still avoid sets of arbitrary complexity, we have to use intervals. This classification result shows the importance of interval-valued fuzzy logics.

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Chapter 1

Introduction

1.1 Need for Fuzzy Logic

It is necessary to describe expert knowledge inside the computer. In many knowledge areas, such as geology, medicine, we rely on experts to make decisions. In each such area, there are a few top experts who have more knowledge than others and who, therefore, make the best decisions. It is desirable to make their expert knowledge available to other experts, so that they can use the expertise of these top experts in their own decision making.

Thus, we need to represent expert knowledge in a computer in such a way that the computer will be able to perform logical reasoning based on this knowledge.

Knowledge representation: successes. Since the early days of computing, there have been computer systems for representing expert knowledge. Probably the most well known idea behind such a representation is the use of Prolog, a special programming language designed for representing knowledge and for reasoning based on this knowledge.

It is necessary to take into account expert uncertainty. Prolog-type systems are a very good tool for describing expert statements which are absolutely correct. For example, if we have a statement that every person participating in a triathlon can swim and we assume this statement to be absolutely correct, then a Prolog-type system would be appropriate to represent this statement and combine it with other absolutely correct statements through reasoning rules to make further deductions.

However, it is not always the case that we have absolutely correct (or absolutely incorrect) statements to base our reasoning on. Instead, statements may be neither absolutely correct nor absolutely incorrect. One such example would be an expert meteorologist predicting that a hurricane will strike a certain area tomorrow morning. This statement introduces uncertainty, since the meteorologist is not absolutely certain that this area will be affected by the hurricane.

In order to adequately represent expert knowledge, it is necessary to take the expert's uncertainty into account.

Fuzzy logic: a formalism for representing expert uncertainty. In a Prolog-type system, every statement is either true or false. In the computer, "true" is usually represented as 1, and "false" is usually represented as 0. The value "true" corresponds to absolute certainty, and value "false" corresponds to the absolute lack of certainty. To represent intermediate degrees of certainty, it is therefore reasonable to use numbers intermediate between 0 and 1: the larger the number, the more we are certain about the given statement.

This idea of using numbers from the interval $[0,1]$ to describe the experts' uncertainty is called a *fuzzy logic*; see, e.g., [2, 5]. Fuzzy logic was first introduced by L. A. Zadeh exactly for this purpose – to formally describe this uncertainty in human reasoning. In fuzzy logic, a person's degree of certainty is described by a number from the interval $[0,1]$, so that absolute certainty in a statement corresponds to 1, absolute certainty in its negation corresponds to 0, and intermediate values, or pure uncertainty, correspond to intermediate degrees of certainty.

1.2 Composite Statements in Fuzzy Logic

Need for composite statements. Our objective is not simply to represent the uncertainty, but also to process this uncertainty.

For example, if we have two statements S_a and S_b, and we know the degree of certainty a in the statement S_a and the degree of certainty b in the statement S_b, then we need to estimate the degree of certainty in a composite statement like $S_a \wedge S_b$ or $S_a \vee S_b$.

Need for an "and" operation. The desired estimate for the degree of certainty of $S_a \wedge S_b$ must depend on the degrees a of S_a and b of S_b. In other words, this desired estimate must be a function of two variables a and b. This function describes the properties of "and" and is therefore sometimes called an "and" function. Let us denote this function by $f_\wedge(a, b)$.

What are the reasonable properties of this function?

First property of an "and" operation: monotonicity. First, if our degree of certainty in one or both of the statements S_a and S_b increases, the resulting degree of belief in $S_a \wedge S_b$ should also increase – or maybe remain the same (but it should not decrease). Thus, the function $f_\wedge(a, b)$ must be a (non-strictly) increasing function of both of its variables: if $a \le a'$ and $b \le b'$, then $f_\wedge(a, b) \le f_\wedge(a', b')$.

Second property: our degree of confidence in $S_a \wedge S_b$ cannot be larger than the degree of confidence in S_a. Second, since $S_a \wedge S_b$ implies S_a, our degree of belief in S_a must be larger than (or at least equal to) the degree of belief in $S_a \wedge S_b$. Formally, we must have $f_\wedge(a, b) \le a$. Similarly, the degree of belief in S_b must be larger than (or at least equal to) the degree of belief in $S_a \wedge S_b$, so we must have $f_\wedge(a, b) \le b$.

Third property: $S_a \wedge S_a$. Third, for every statement S_a, "S_a and S_a" means the same as S_a. In terms of the function $f_\wedge$, this means that $f_\wedge(a, a) = a$ for all a.

Resulting "and" operation: derivation and result. Let us show how these three requirements lead to a definition selection of the estimation function $f_\wedge(a, b)$. Let us first consider the case when $a \le b$. In this case, due to monotonicity, we have $f_\wedge(a, a) \le f_\wedge(a, b)$.

Since $f_\wedge(a, a) = a$, we thus conclude that $a \le f_\wedge(a, b)$. On the other hand, we know that $f_\wedge(a, b) \le a$. So, we conclude that $f_\wedge(a, b) = a$.

Similarly, in the case when $a > b$, we conclude that $b = f_\wedge(b, b) \le f_\wedge(a, b) \le b$, hence $f_\wedge(a, b) = b$. Both cases $a \le b$ and $a > b$ can be described by a single formula $f_\wedge(a, b) = \min(a, b)$.

Need for an "or" operation. The desired estimate for the degree of certainty of $S_a \vee S_b$ must also depend on the degrees a of S_a and b of S_b. In other words, this desired estimate must be a function of two variables a and b. Let us denote this function by $f_\vee(a, b)$. Similarly to the "and" case, this function must satisfy the following three properties.

First property of an "or" operation: monotonicity. First, if our degree of certainty in one or both of the statements S_a and S_b increases, the resulting degree of belief in $S_a \vee S_b$ should also increase – or maybe remain the same (but it should not decrease). Thus, the function $f_\vee(a, b)$ must also be a (non-strictly) increasing function of both of its variables.

Second property: our degree of confidence in $S_a \vee S_b$ cannot be smaller than the degree of confidence in S_a. Second, since S_a implies $S_a \vee S_b$, our degree of belief in S_a must be smaller than (or at least equal to) the degree of belief in $S_a \vee S_b$. Formally, we must have $f_\vee(a, b) \ge a$. Similarly, the degree of belief in S_b must be smaller than (or at least equal to) the degree of belief in $S_a \vee S_b$, so we must have $f_\vee(a, b) \ge b$.

Third property: $S_a \vee S_a$. Third, for every statement S_a, "S_a or S_a" means the same as S_a. In terms of the function $f_\vee$, this means that $f_\vee(a, a) = a$ for all a.

Resulting "or" operation: derivation and result. Let us show how these three requirements lead to a definition selection of the estimation function $f_\vee(a, b)$. Let us first consider the case when $a \le b$. In this case, due to monotonicity, we have $f_\vee(a, b) \le f_\vee(b, b)$.

Since $f_\vee(b,b) = b$, we thus conclude that $f_\vee(a,b) \le b$. On the other hand, we know that $b \le f_\vee(a,b)$. So, we conclude that $f_\vee(a,b) = b$.

Similarly, in the case when $a > b$, we conclude that $a \le f_\vee(a,b) \le f_\vee(a,a) = a$, hence $f_\vee(a,b) = a$. Both cases $a \le b$ and $a > b$ can be described by a single formula $f_\vee(a,b) = \max(a,b)$.

"And" and "or" operations: conclusion. In line with this reasoning, we usually estimate the degree of certainty in composite statements $S_a \wedge S_b$ and $S_a \vee S_b$ as, correspondingly, $a \wedge b \stackrel{\text{def}}{=} \min(a,b)$ and $a \vee b \stackrel{\text{def}}{=} \max(a,b)$.

1.3 Fuzzy Logic: Re-Scalings

The existence of different scales. The main purpose of fuzzy logic is to quantify uncertainty.

In quantification, usually, there are many different scales for representing the same quantity. For example:

- a person's height can be measured in inches or in centimeters;
- a temperature can be described in the Fahrenheit scale (F) or in Celsius scale (C);
- the energy of an earthquake can be described in the energy units, or in the logarithmic (Richter) scale – as the logarithm of this energy.

In different scales, you use different methods and thus, arrive at the different numbers for representing the same quantity. For example, if we use a centimeter-marked ruler, we get a person's height in centimeters, and if we use an inch-marked ruler, we get the same height in inches. These two numerical values differ by a factor of 2.54. Similarly, if we measure a temperature by two different thermometers, we get two different values for representing the same weather conditions: 0°C is the same as 32°F.

Similarly, in fuzzy logic, there are different ways to quantify uncertainty which lead, in general, to different numerical values; see, e.g., [2, 5].

Need for re-scaling. Re-scaling means translating a number describing some quantity in one scale into the numerical value of this same quantity in a different scale.

For example, to re-scale the height in cm into a height in inches, we must divide the original value by 2.54. Vice versa, to re-scale the height in inches into a height in cm, we must multiply the corresponding numerical value by 2.54.

Similarly, to translate a temperature t_C in C into a temperature t_F in F, we must use the re-scaling formula $t_F = 32 + 1.8 \cdot t_C$. To re-scale the earthquake energy into the Richter scale, we must take the logarithm of the original energy value, etc.

For each re-scaling, there is usually an inverse re-scaling: to a re-scaling from cm to inches, the reverse re-scaling transforms the corresponding numerical values back into cm.

Reasonable re-scalings for fuzzy logic: description. What are the reasonable scales for fuzzy logic?

In our description of fuzzy degrees, we assumed that 1 represents absolute certainty, 0 represents absolute certainty that a statement is false, and values between 0 and 1 represent intermediate degrees of certainty – the larger our degree of certainty, the larger the corresponding fuzzy value.

It is therefore reasonable to consider re-scalings $\varphi : [0,1] \to [0,1]$ which preserve these properties, i.e., 1-1 mappings which satisfy the following three properties:

- $\varphi(0) = 0$,
- $\varphi(1) = 1$, and
- for every two values a and b, $a < b$ if and only if $\varphi(a) < \varphi(b)$.

We also want to make sure that there exists an inverse function which satisfies the same properties. In mathematical terms, a reasonable re-scaling can thus be defined as bijection

(1-1 onto mapping) which is strictly monotonic.

Observation: reasonable re-scaling preserve "and" and "or" operations. It is easy to check that every monotonic bijection φ preserves the operations $a \wedge b$ and $a \vee b$ in the sense that for every $a, b \in [0, 1]$, we have $\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$ and $\varphi(a) \vee \varphi(b) = \varphi(a \vee b)$.

These two equalities clearly hold when $a = b$. To prove these two statements, let us consider two remaining cases: when $a < b$ and when $a > b$.

If $a < b$, then $a \wedge b = \min(a, b) = a$. Due to monotonicity, we have $\varphi(a) < \varphi(b)$, hence $\varphi(a) \wedge \varphi(b) = \min(\varphi(a), \varphi(b)) = \varphi(a)$. So, in this case, we have $\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$. Similarly, when $a < b$, we have $a \vee b = \max(a, b) = b$. Due to monotonicity, we have $\varphi(a) < \varphi(b)$, hence $\varphi(a) \vee \varphi(b) = \max(\varphi(a), \varphi(b)) = \varphi(b)$. So, in this case, we have $\varphi(a) \vee \varphi(b) = \varphi(a \vee b)$.

If $a > b$, then $a \wedge b = \min(a, b) = b$. Due to monotonicity, we have $\varphi(a) > \varphi(b)$, hence $\varphi(a) \wedge \varphi(b) = \min(\varphi(a), \varphi(b)) = \varphi(b)$. So, in this case, we have $\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$. Similarly, when $a > b$, we have $a \vee b = \max(a, b) = a$. Due to monotonicity, we have $\varphi(a) > \varphi(b)$, hence $\varphi(a) \vee \varphi(b) = \max(\varphi(a), \varphi(b)) = \varphi(a)$. So, in this case, we also have $\varphi(a) \vee \varphi(b) = \varphi(a \vee b)$.

Observation: "reasonable" re-scalings can be defined as bijections which preserve "and" and "or" operations. It is possible to show the contraposite to what we have just proven: that every bijection which preserves the operations $\wedge$ and $\vee$ is monotonic.

Indeed, let us assume that we have a bijection $\varphi : [0, 1] \to [0, 1]$ for which $\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$. If $a < b$, then $a \wedge b = \min(a, b) = a$, and the above property takes the form $\varphi(a) \wedge \varphi(b) = \varphi(a)$, i.e., the form $\min(\varphi(a), \varphi(b)) = \varphi(a)$. This means that $\varphi(a)$ is the smaller of the two numbers, i.e., $\varphi(a) \le \varphi(b)$. The values $\varphi(a)$ and $\varphi(b)$ cannot be equal, since $a \ne b$ and φ is a bijection. Thus, we must have $\varphi(a) < \varphi(b)$. Since this is true for all $a < b$, this means that the function φ is indeed strictly monotonic.

Reformulation in terms of automorphisms. In mathematical terms, transformations which preserve certain operations are called *automorphisms* of the corresponding structure, and the set of all such transformations is called its *automorphism group*. In these terms, the automorphism group of the structure $([0,1], \wedge, \vee)$ is the group of all strictly monotonic continuous functions from $[0,1]$ to $[0,1]$ for which $\varphi(0) = 0$ and $\varphi(1) = 1$.

1.4 From Single-Valued Fuzzy Logic to Interval-Valued and Set-Valued Ones

Need for a fuzzy logic: reminder. As we have mentioned earlier, experts are usually not fully certain about their statements. In traditional fuzzy logic, the expert's degree of certainty in each of his or her statements is described by a number from the interval $[0,1]$.

Need to go beyond traditional fuzzy logic. The need for fuzzy logic comes from the expert's uncertainty: an expert is not 100% sure about about the exact value of, say, a temperature. So, instead of providing the system with the exact value of the temperature, the expect provides several possible values – e.g., an interval or a set of possible values.

For example, an expert can say that the temperature is between 80 and 90. In this case, the set of possible values is the interval $[80, 90]$.

An expert may have different degrees of confidence in different values from this interval. For example, an expert can have more belief in values from 83 to 87 than in values which are closer to the endpoints 80 and 90 of this interval. To quantify these different degrees of certainty, we use fuzzy logic. In effect, we ask the expert to quantify his or her degree of uncertainty by supplying us with a number from the interval $[0,1]$ which describes this degree of belief.

However, just as an expert cannot pinpoint a single value of the temperature, this same expert cannot pinpoint a single value as describing his or her degree of certainty. Just like an expert is usually more comfortable providing an interval (or, more generally, a set) of possible values of temperature, this same expert is more comfortable providing an interval (or, more generally, a set) of possible values of degrees of certainty.

In other words, to describe the expert's degree of certainty in a given statement, instead of a single value $a \in [0, 1]$, it is more adequate to provide a *set* $A \subseteq [0, 1]$ of possible values.

"And" and "or" operations on set-valued degrees of uncertainty. There is a natural extension of operations $\wedge$ and $\vee$ to such sets. Indeed, a set A means that all values $a \in A$ are possible, B means that all the values $b \in B$ are possible; so the set $A \wedge B$ of possible values of $a \wedge b$ is formed by all the values $a \wedge b$ where $a \in A$ and $b \in B$:

$$A \wedge B \stackrel{\text{def}}{=} \{a \wedge b : a \in A, b \in B\}. \tag{1.1}$$

Similarly,

$$A \vee B \stackrel{\text{def}}{=} \{a \vee b : a \in A, b \in B\}. \tag{1.2}$$

1.5 Formulation of the Practical Problem: Which Sets Should We Use?

Traditionally, intervals are mainly used. In many applications, researchers have been successfully using *intervals* of possible values of degree of uncertainty; see, e.g., [2, 4, 5].

In principle, it is possible to use more general sets. However, it is possible to consider more general sets as well [6]: e.g., a set $\{0, 1\}$ consisting of two elements 0 and 1 is not an interval.

Formulation of the problem. A natural question, whose answer is the objective of this thesis is:

Which sets should we consider?

To answer this question, we need to describe it in precise terms.

1.6 Towards Formulating the Problem in Precise Terms

We want an extension. Since we are talking about *extensions* of traditional fuzzy logic, it is reasonable to require that the desired class of sets $\mathcal{S}$ contain all one-element sets (corresponding to traditional fuzzy values).

We want invariance. It is also reasonable to assume that the class $\mathcal{S}$ is invariant under automorphisms of traditional fuzzy logic.

In precise terms, if S is a possible set (i.e., if $S \in \mathcal{S}$), and $\varphi(x)$ is a strictly increasing continuous function with $\varphi(0) = 0$ and $\varphi(1) = 1$, then the image $\varphi(S) = \{\varphi(s) : s \in S\}$ should also be a possible set - i.e., we should have $(\varphi(S) \in \mathcal{S})$.

We want closure under $\wedge$ and $\vee$ Another reasonable requirement is that the class $\mathcal{S}$ be closed under naturally defined operations $\wedge$ and $\vee$.

In precise terms, if A and B are possible sets (i.e., if $A, B \in \mathcal{S}$), then the sets $A \wedge B$ and $A \vee B$ should also be possible, i.e., we should have $A \wedge B \in \mathcal{S}$ and $A \vee B \in \mathcal{S}$.

It is sufficient to consider closed sets. There is one more property that is natural to assume. If, according to a set $S \in \mathcal{S}$, the values $s_1, s_2, \ldots, s_k, \ldots$ are all possible, and the sequence s_k converges to a certain number s, then no matter how accurately we compute s, we will always find a possible number s_k that is indistinguishable from s. Therefore, it is natural to assume that this limit value s is also possible.

In other words, it is natural to assume that every set $S \in \mathcal{S}$ contains all its limit points, i.e., that it is a *closed* set.

It is sufficient to consider closed classes of sets. A similar requirement can be formulated for different sets $S \in \mathcal{S}$. Indeed, on the class of all bounded closed sets, there is a natural metric – Hausdorff metric $d_H(S, S')$. This metric is defined as the smallest $\varepsilon > 0$ for which S is contained in the ε-neighborhood of S' and S' is contained in the

ε-neighborhood of S, i.e., for which

$$\forall s \in S\, \exists s' \in S'\, (d(s,s') \le \varepsilon) \,\wedge\, \forall s' \in S'\, \exists s \in S\, (d(s,s') \le \varepsilon),$$

where $d(s,s') = |s - s'|$ is the standard distance between the points on the real line.

Comment. It is worth mentioning that the Hausdorff distance between two degenerate intervals, i.e., intervals $[s,s] = \{s\}$ and $[s',s'] = \{s'\}$, is exactly the Euclidean distance between the corresponding two points:

$$d_H([s,s],[s',s']) = d(s,s').$$

In general, the Hausdorff distance between two intervals $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$ is equal to

$$d_H([\underline{a},\overline{a}],[\underline{b},\overline{b}]) = \max(d(\underline{a},\underline{b}), d(\overline{a},\overline{b})) = \max(|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|).$$

For example, for $A = [-1,1]$ and $B = [0,3]$, we get

$$d_H(A,B) = \max(d(-1,0), d(1,3)) = \max(|(-1)-0|, |1-3|) = 2;$$

for $A = [2,3]$ and $B = [2,6]$, we get

$$d_H(A,B) = \max(d(2,2), d(3,6)) = \max(|2-2|, |3-6|) = 3.$$

Informally, d_H means that if $d_H(S,S') = 0$, and we only know the values $s \in S$ and $s' \in S'$ with accuracy ε, then we cannot distinguish between the sets S and S'.

So, if the sets $S_1, S_2, \ldots, S_k, \ldots$ are all possible (i.e., $S_i \in \mathcal{S}$), and the sequence of sets S_k converges to a certain set S (i.e., $d_H(S_k,S) \to 0$), then no matter how accurately we compute the values, we will always find a set S_k that is indistinguishable from the set S (and possible). Therefore, it is natural to assume that this limit set S is also possible. In other words, it is natural to assume that the class $\mathcal{S}$ contains all its limit points, i.e., that it is a *closed* class under the Hausdorff metric.

We are now ready to formulate the main classification result.

Chapter 2

Classification Theorems

2.1 Main Definition

Definition 1 *A class $\mathcal{S}$ of closed non-empty subsets of the interval $[0,1]$ is called a* set-valued extension of fuzzy logic *if it satisfies the following conditions:*

(i) the class $\mathcal{S}$ contains all 1-element sets $\{s\}$, $s \in [0,1]$;

(ii) the class $\mathcal{S}$ is closed under "and" and "or" operations (1.1) and (1.2);

(iii) the class $\mathcal{S}$ is closed under arbitrary automorphisms of $([0,1], \wedge, \vee)$, i.e., if $S \in \mathcal{S}$ and $\varphi(x)$ is a strictly increasing function for which $\varphi(0) = 0$ and $\varphi(1) = 1$, then $\varphi(S) \in \mathcal{S}$; and

(iv) the class $\mathcal{S}$ is closed under Hausdorff metric.

2.2 Main Result

Theorem 1 *Every set-valued extension of fuzzy logic coincides with one of the following three classes:*

- *the class of all one-point sets $\{s\}$;*
- *the class of all subintervals $[\underline{s}, \overline{s}] \subseteq [0,1]$ of the interval $[0,1]$;*
- *the class of all closed subsets S of the interval $[0,1]$.*

Comments.

- This result shows that under reasonable conditions, every set-valued extension of fuzzy logic coincides either with the traditional fuzzy logic, or with interval-valued fuzzy logic, or with the class of all closed subsets of the interval $[0, 1]$. So, if want to go beyond traditional single-valued fuzzy sets and do not want to consider arbitrarily complex closed sets, we must use intervals. This classification result shows the importance of interval-valued fuzzy logics.

- All the proofs are placed in Chapter 3.

2.3 First Auxiliary Result: No Need to Require Single-Valued Fuzzy Values

Since single-valued fuzzy values are probably un-realistic, it may not be necessary to require that one-point sets belong to the class $\mathcal{S}$. It turns out that for our classification, it is not necessary to require one-point sets, it is sufficient to require that there is at least one set $S \in \mathcal{S}$ which corresponds to "pure uncertainty", i.e., does not contain 0 and does not contain 1.

Theorem 2 *Every class $\mathcal{S}$ of closed non-empty subsets of the interval $[0, 1]$ which satisfies the condition*

(i′) the class $\mathcal{S}$ contains a set S for which $0 \notin S$ and $1 \notin S$,

and conditions (ii)-(iv) has one of the three forms from Theorem 1.

2.4 Second Auxiliary Result: A General Classification Result

A natural question is: What happens if the opposite to (i′) is true, i.e., if every set $S \in \mathcal{S}$ contains either 0 or 1? In this case, several other classes are possible:

Theorem 3 *Every class $\mathcal{S}$ of closed non-empty subsets of the interval $[0,1]$ which satisfies the condition*

(i″) every set $S \in \mathcal{S}$ contains either 0 or 1,

and conditions (ii)-(iv), is a union of one or more of the following classes:

- *the class consisting of a single set $\{0\}$;*
- *the class consisting of a single set $\{1\}$;*
- *the class consisting of a single interval $[0,1]$;*
- *the class consisting of a single set $\{0,1\}$;*
- *the class I_0 of all subintervals $S \subseteq [0,1]$ which contain 0, i.e., the class of all subintervals of the type $[0,\overline{s}]$;*
- *the class I_1 of all subintervals $S \subseteq [0,1]$ which contain 1, i.e., the class of all subintervals of the type $[\underline{s},1]$;*
- *the class I_0^+ of all sets $S \subseteq [0,1]$ of the type $[0,\overline{s}] \cup \{1\}$;*
- *the class I_1^+ of all sets $S \subseteq [0,1]$ of the type $[\underline{s},1]$;*
- *the class I_{01} of all sets $S \subseteq [0,1]$ of the type $[0,\underline{s}] \cup [\overline{s},1]$;*
- *the class C_0 of all closed subsets $S \subseteq [0,1]$ which contain 0;*
- *the class C_1 of all closed subsets $S \subseteq [0,1]$ which contain 1;*
- *the class C_{01} of all closed subsets $S \subseteq [0,1]$ which contain both 0 and 1.*

Comment. Here, one of the cases is when we have a 3-valued logic (true = 1, false = 0, and unknown = $[0, 1]$) or its sublogic (including the case of classical logic $\mathcal{S} = \{\{0\}, \{1\}\}$). In all other cases, we have either intervals or arbitrarily complex closed set. So, here too, if we do not want arbitrary complex sets, we must restrict ourselves to intervals.

2.5 From Set-Valued to Type-2 Fuzzy Logic

Set-values fuzzy logics are a particular case of general type-2 fuzzy sets, in which a degree of certainty is itself a fuzzy set. In particular, instead of intervals, it is reasonable to consider *fuzzy numbers* as fuzzy sets, i.e., membership functions which increase to 1 and then decrease back to 0. An important case is *strictly monotonic* fuzzy numbers (e.g., triangular ones) in which the membership function continuously strictly increase to 1 and then continuously strictly decreases back to 0.

It is worth mentioning that every two such functions can be transformed into each other by an appropriate automorphism $\varphi : [0, 1] \to [0, 1]$, and that very other fuzzy number can be represented as a limit of strictly monotonic ones. Thus, if a class $\mathcal{S}$ of fuzzy sets contains *at least one* strictly increasing fuzzy number and that it is invariant under automorphisms and closed (in the sense of the appropriately defined Hausdorff metric), that $\mathcal{S}$ should contain *all* fuzzy numbers.

Chapter 3

Proofs

3.1 Proof of Theorem 1

1°. Let $\mathcal{S}$ be a set-valued extension of fuzzy logic. We want to prove that $\mathcal{S}$ coincides with one of the three classes described in the formulation of the theorem.

By Definition 1 of a set-valued extension of fuzzy logic, the class $\mathcal{S}$ contains all one-element sets $\{s\}$ for all the values $s \in [0, 1]$. Let us first consider the case when $\mathcal{S}$ coincides with the class of all one-element sets, i.e., when $\mathcal{S}$ consists only of one-element sets. In this case, we have the first class from the formulation of Theorem 1.

So, to complete the proof, it is sufficient to consider only the case when not all sets from the class $\mathcal{S}$ are one-element sets, i.e., when the class $\mathcal{S}$ contains at least one set which has two or more points.

In the following proof, we will consider two possible subcases of this case:

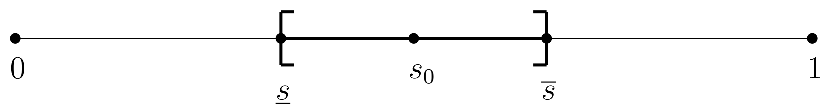
- first, we will consider the case when all sets from the class $\mathcal{S}$ are intervals;
- then, we will consider the remaining case when some sets from the class $\mathcal{S}$ are not intervals.

2°. Let us first consider the case where all sets from the class $\mathcal{S}$ are intervals. We will prove that in this case, the class $\mathcal{S}$ coincides with the class of all subintervals in $[0, 1]$.

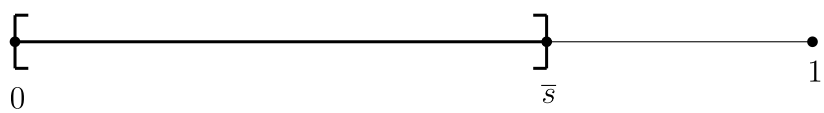
Our case is a subcase of the general case in which the class $\mathcal{S}$ contains at least one set which has two or more points. Let us select one such set and denote it by S_0. Since all sets from the class $\mathcal{S}$ are intervals, this means that the set S_0 (which has two or more points) is also an interval, i.e., it has the form $[\underline{s}, \overline{s}]$ for some real numbers $\underline{s} \le \overline{s}$.

Since the set S_0 is not a one-element set, this interval must be non-degenerate, i.e., we must have $\underline{s} < \overline{s}$.

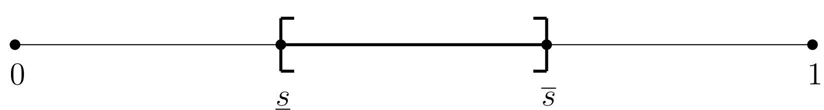
Let us denote the midpoint of this non-degenerative interval by $s_0 = \dfrac{\underline{s} + \overline{s}}{2}$.



2.1°. Let us first prove that the class $\mathcal{S}$ contains the interval $[0, \overline{s}]$.

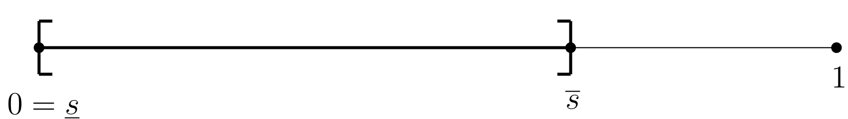


By definition of the values $\underline{s}$ and $\overline{s}$, the class $\mathcal{S}$ contains the interval $S_0 = [\underline{s}, \overline{s}]$.

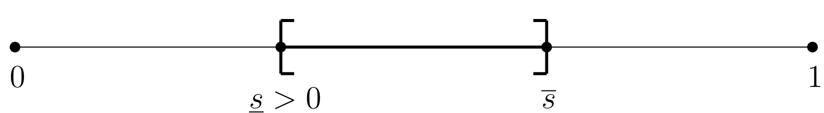


There are two possible situations:

- it is possible that that $\underline{s} = 0$;

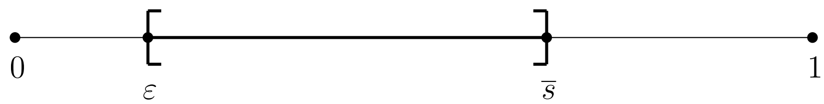


- it is also possible that $\underline{s} > 0$.

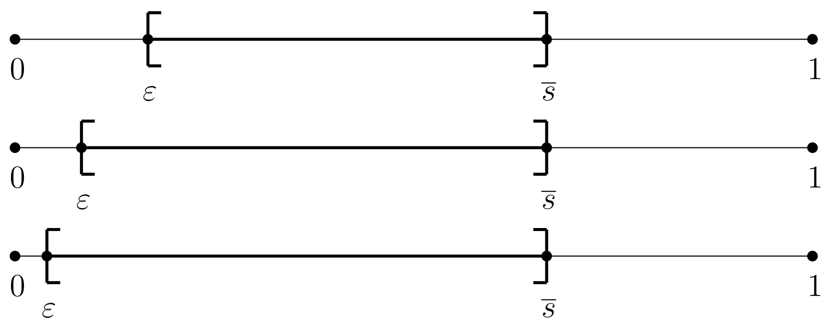


In the first situation $\underline{s} = 0$, we have $[\underline{s}, \overline{s}] = [0, \overline{s}]$. Since the interval $[\underline{s}, \overline{s}]$ belongs to $\mathcal{S}$, we thus conclude that $[0, \overline{s}] \in \mathcal{S}$.

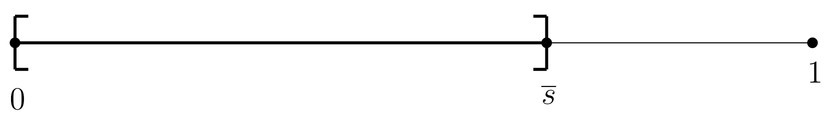
Let us now consider the situation when $\underline{s} > 0$. In this situation, to prove that $[0, \overline{s}] \in \mathcal{S}$, we will use the fact that the class $\mathcal{S}$ is closed, i.e., that it contains the limits of all its sets. So, if we manage to prove that for every ε from the open interval $(0, \underline{s})$, we have $[\varepsilon, \overline{s}] \in \mathcal{S}$,



then for $\varepsilon \to 0$,



we will be able to conclude that $[\varepsilon, \overline{s}] \to [0, \overline{s}]$



and thus, that $[0, \overline{s}] \in \mathcal{S}$.

To prove that $[\varepsilon, \overline{s}] \in \mathcal{S}$ for every $\varepsilon \in (0, \underline{s})$, we will use another property of a set-valued extension of fuzzy logic: that this extension must be closed under strictly increasing functions $\varphi(x)$.

To use this property, for every ε from 0 to $\underline{s}$, let us construct a piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(\underline{s}) = \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, and $\varphi_\varepsilon(1) = 1$. In general, a linear function $\varphi(x)$ for which $\varphi(\underline{a}) = \underline{b}$ and $\varphi(\overline{a}) = \overline{b}$ for some $\underline{a} < \overline{a}$ can be described by the following formula:

$$\varphi(x) = \underline{b} + \frac{\overline{b} - \underline{b}}{\overline{a} - \underline{a}} \cdot (x - \underline{a}).$$

Thus, our piece-wise linear function $\varphi_\varepsilon(x)$ has the following form:

- for $0 \le x \le \underline{s}$, we have

$$\varphi_\varepsilon(x) = \frac{\varepsilon}{\underline{s}} \cdot x;$$

- for $\underline{s} \le x \le s_0$, we have

$$\varphi_\varepsilon(x) = \varepsilon + \frac{s_0 - \varepsilon}{s_0 - \underline{s}} \cdot (x - \underline{s});$$

- finally, for $s_0 \le x \le 1$, we have

$$\varphi_\varepsilon(x) = s_0 + \frac{1 - s_0}{1 - s_0} \cdot (x - s_0) = s_0 + (x - s_0) = x.$$

For the points $0 < \underline{s} < 1$, the values of this function are increasing: $0 < \varepsilon < 1$. Thus, the piece-wise linear function $\varphi_\varepsilon(x)$ is strictly increasing. Since this function is strictly increasing,

- its smallest possible value on the interval $[\underline{s}, \overline{s}]$ is attained at the smallest possible values $s = \underline{s}$, and

- its largest possible value on the interval $[\underline{s}, \overline{s}]$ is attained at the largest possible values $s = \overline{s}$.

Thus, $\varphi_\varepsilon([\underline{s}, \overline{s}]) = [\varepsilon, \overline{s}]$. By the property of the class $\mathcal{S}$, we can now conclude that $[\varepsilon, \overline{s}] \in \mathcal{S}$.

When we allow ε to approach 0 ($\varepsilon \to 0$), then $[\varepsilon, \overline{s}] \to [0, \overline{s}]$. Since class $\mathcal{S}$ is closed, we can conclude that $[0, \overline{s}] \in \mathcal{S}$.

2.2°. Let us now prove that the class $\mathcal{S}$ contains the interval $[0, 1]$.

We have just proved that the class $\mathcal{S}$ contains the interval $[0, \overline{s}]$. There are two possible situations:

- it is possible that that $\overline{s} = 1$;

- it is also possible that $\overline{s} < 1$.

In the first situation $\overline{s} = 1$, we have $[0, \overline{s}] = [0, 1]$. Since the interval $[0, \overline{s}]$ belongs to $\mathcal{S}$, we thus conclude that $[0, 1] \in \mathcal{S}$.

Let us now consider the situation when $\overline{s} < 1$. In this situation, to prove that to prove that $[0, 1] \in \mathcal{S}$, we will also use the fact that the class $\mathcal{S}$ is closed, i.e., that it contains the limits of all its sets. So, if we manage to prove that for every ε from the open interval $(0, 1 - \overline{s})$, we have $[0, 1 - \varepsilon] \in \mathcal{S}$, then for $\varepsilon \to 0$, we will be able to conclude that $[0, 1 - \varepsilon] \to [0, 1]$ and thus, that $[0, 1] \in \mathcal{S}$.

To prove that $[0, 1 - \varepsilon] \in \mathcal{S}$ for every $\varepsilon \in (0, \underline{s})$, we will use the same property of a set-values extension of fuzzy logic used previously: that this extension must be closed under strictly increasing functions $\varphi(x)$.

For every $\varepsilon \in (0, 1 - \overline{s})$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(\overline{s}/2) = \overline{s}/2$, $\varphi_\varepsilon(\overline{s}) = 1 - \varepsilon$, and $\varphi_\varepsilon(1) = 1$. Thus, our piece-wise linear function $\varphi_\varepsilon(x)$ has the following form:

- for $0 \le x \le \overline{s}/2$, we have
$$\varphi_\varepsilon(x) = \frac{\overline{s}/2}{\overline{s}/2} \cdot x = x;$$
- for $\overline{s}/2 \le x \le \overline{s}$, we have
$$\varphi_\varepsilon(x) = \overline{s}/2 + \frac{(1 - \varepsilon) - \overline{s}/2}{\overline{s} - \overline{s}/2} \cdot (x - \overline{s}/2);$$
- finally, for $\overline{s} \le x \le 1$, we have
$$\varphi_\varepsilon(x) = (1 - \varepsilon) + \frac{1 - (1 - \varepsilon)}{1 - \overline{s}} \cdot (x - \overline{s}).$$

For the points $0 < \overline{s}/2 < \overline{s} < 1$, the values of this function are increasing:

$$0 < \overline{s}/2 < 1 - \varepsilon < 1.$$

Thus, the piece-wise linear function $\varphi_\varepsilon(x)$ is strictly increasing. Since this function is strictly increasing,

- its smallest possible value on the interval $[0, \overline{s}]$ is attained at the smallest possible values $s = 0$, and
- its largest possible value on the interval $[0, \overline{s}]$ is attained at the largest possible values $s = \overline{s}$.

Thus, $\varphi_\varepsilon([0, \overline{s}]) = [0, 1 - \varepsilon]$. By the property of the class $\mathcal{S}$, we can now conclude that $[0, 1 - \varepsilon] \in \mathcal{S}$.

When we allow ε to approach 0 ($\varepsilon \to 0$), then $[0, 1 - \varepsilon] \to [0, 1]$. Since class $\mathcal{S}$ is closed, we can conclude that $[0, 1] \in \mathcal{S}$.

2.3°. We are providing our proof within an assumption that all the sets from the family $\mathcal{S}$ are intervals. Under this assumption, we want to prove that the family $\mathcal{S}$ coincides with the class of all subintervals of the interval $[0, 1]$.

We have already shown that $[0, 1] \subseteq \mathcal{S}$. To complete the proof under this assumption, we will now prove that the class $\mathcal{S}$ can contain an arbitrary interval $[a, b] \in [0, 1]$.

By definition, the class $\mathcal{S}$ contains all 1-point sets. In particular, the class $\mathcal{S}$ contains the 1-element sets $\{a\}$ and $\{b\}$. According to the property (ii) from Definition 1, the class $\mathcal{S}$ is closed under $\wedge$, i.e., for every two sets $S_1 \in \mathcal{S}$ and $S_2 \in \mathcal{S}$, the set $S_1 \wedge S_2$ should also belong to the class $\mathcal{S}$. Similarly, the class $\mathcal{S}$ is closed under $\vee$, i.e., for every two sets $S_1 \in \mathcal{S}$ and $S_2 \in \mathcal{S}$, the set $S_1 \vee S_2$ also belongs to the class $\mathcal{S}$.

Since we have proved that $[0, 1] \in \mathcal{S}$ and that $\{a\} \in \mathcal{S}$, we thus conclude that $[0, 1] \vee \{a\} \in \mathcal{S}$. One can easily check that

$$[0, 1] \vee \{a\} = \{\max(s, a) : s \in [0, 1]\} = [a, 1].$$

Similarly, knowing that $a \le b$, we conclude that $[a, 1] \wedge \{b\} = [a, b] \in \mathcal{S}$.

So, the class $\mathcal{S}$ contains all intervals. Since we are in the case when all its elements are intervals, the class $\mathcal{S}$ thus coincides with the class of all subintervals of the interval $[0, 1]$.

3°. So far, we have considered the case when all the elements of the class $\mathcal{S}$ are intervals. We have shown that in this case, the class $\mathcal{S}$ coincides with either the class P of all one-point sets or with the class I of all subintervals of the interval $[0, 1]$.

Let us now consider the remaining case when the class $\mathcal{S}$ contains a closed set S which is not an interval.

3.1°. Let us prove that in this case, the class $\mathcal{S}$ contains the set $\{0, 1\}$.

In this proof, we will use the notions of the infimum (greatest lower bound) inf and supremum (least upper bound) sup. Let us recall the definitions of these notions.

A number x is called a *lower bound* of a set S if it is smaller than or equal to all the elements of this set. The infimum $\inf S$ is defined as the greatest lower bound, i.e., as a lower bound which is larger than or equal to any other lower bound. In other words, the infimum is the greatest of all the numbers which are less than or equal to all the numbers from the set S.

Similarly, a number x is called an *upper bound* of a set S if it is greater than or equal to all the elements of this set. The supremum $\sup S$ is defined as the least upper bound, i.e., as an upper bound which is smaller than or equal to any other upper bound. In other words, the supremum is the smallest of all the numbers which are greater than or equal to all the numbers from the set S.

It is known that every bounded set has the infimum and the supremum. Since we only consider subsets of the interval $[0, 1]$, all these subsets are bounded and therefore each of them has the infimum and the supremum.

Let $S \in \mathcal{S}$ be a closed set which is not an interval. As we have just mentioned, this set has the infimum and the supremum. Let us denote the infimum of this set by $s^- \stackrel{\text{def}}{=} \inf S$, and its supremum by $s^+ \stackrel{\text{def}}{=} \sup S$.

By definition of the infimum, the value $s^- = \inf S$ is a lower bound for the set S, i.e., every element from the set S is greater than or equal to s^-. By definition of the supremum, the value $s^+ = \sup S$ is an upper bound for the set S, i.e., every element from the set S is smaller than or equal to s^+. Thus, every element $s \in S$ satisfies the property $s^- \le s \le s^+$, i.e., every element s belongs to the interval $[s^-, s^+]$. Thus, we conclude that the set S is contained in the interval $[s^-, s^+]$, i.e., $S \subseteq [s^-, s^+]$.

We assumed that the set S is not an interval. This means that this set is different from every interval. In particular, this means that the set S is different from the interval $[s^-, s^+]$, i.e., that $S \neq [s^-, s^+]$.

By definition, the two sets are equal if every element of the first set belongs to the second one, and every element of the second set belongs to the first one. For the set S and $[s^-, s^+]$, we know that $S \subseteq [s^-, s^+]$, i.e., that every element of the set S belongs to the interval $[s^-, s^+]$. Thus, the fact that these two sets are different means that not every element from the interval $[s^-, s^+]$ belongs to the set S. In other words, there must exist a point $s_0 \in [s^-, s^+]$ which is not contained in the set S.

By definition, all the sets from the class $\mathcal{S}$ are closed. Thus, our set S is also closed. In other words, for every sequence of elements from this set, its limit point is also contained in this set S. It is known that both infimum and supremum of a set are its limit points. Since the set S is closed, it must contain its limit points s^- and s^+. Thus, since s_0 is not contained in S, the point s_0 must be different from these points s^- and s^+ – otherwise, it will be contained in S. In other words, the point $s_0 \notin S$ must be strictly between s^- and s^+: $s^- < s_0 < s^+$.

From this, we conclude that $0 < s_0 < 1$.

It is known that a complement to a closed set is open. A set is called open if with every element, it contains an open neighborhood of this element, i.e., in this case, an open interval which contains this element. In particular, the complement to the closed set S is open. The value s_0 does not belong to the set S, it therefore belongs to its complement – which is an open set. So, since $s_0 \notin S$, there exists a whole open interval $(\underline{s}, \overline{s})$ containing the point s_0 which is contained in the complement to S – i.e., which has no common point with S.

Let us illustrate this idea on two simple examples. The first example is the set $S = [0, 0.2] \cup [0.4, 1]$. For this set, the infimum s^- is equal to 0, and the supremum s^+ is equal to 1. So, the set S is contained in the interval $[0, 1]$, but it is different from this interval. For example, the set S does not contain the point $s_0 = 0.3$ which is an element of $[0, 1]$. The point $s_0 = 0.3$ belongs to the open complement of the set S, and thus, there is a neighborhood of this point – e.g., the interval $(0.25, 0.35)$ – which is contained in the complement. So, in this case, we have an open interval $(0.25, 0.35)$ which has no common point with the set S.

Another example is the set $S = \{0.5, 0.7\}$. For this set, $s^- = 0.5$, $s^+ = 0.7$, and for $s_0 = 0.6$, we have $s_0 \in [s^-, s^+] = [0.5, 0.7]$ but $s_0 \notin S$. Thus, there exists an open interval which contains $s_0 = 0.6$ and which is completely outside the set S – e.g., the interval $(0.59, 0.61)$.

In general, we have constructed an open interval $(\underline{s}, \overline{s})$ which is completely outside the set S. Most properties of the set S are formulated in terms of closed set; so, let us construct an auxiliary closed interval $[t^-, t^+]$ with the same property – that this closed interval does not have any common points with the set S. To construct such a closed interval, it is sufficient to take a closed interval which is strictly inside the open interval $(\underline{s}, \overline{s})$. For example, we can take $t^- \stackrel{\text{def}}{=} \dfrac{\underline{s} + s_0}{2}$ and $t^+ \stackrel{\text{def}}{=} \dfrac{s_0 + \overline{s}}{2}$. In this case, $s^- \le \underline{s} < t^- < s_0 < t^+ < \overline{s} \le s^+$, and $S \cap [t^-, t^+] = \emptyset$.

Since the set $S \subseteq [0, 1]$ does not have any elements from the interval $[t^-, t^+]$, we can thus conclude that $S \subseteq [0, t^-] \cup [t^+, 1]$, i.e., every element of the set S is contained either in the interval $[0, t^-]$, or in the interval $[t^+, 1]$. In particular, since $s^- \le \underline{s} < t^-$ and $t^+ < \overline{s} \le s^+$, we conclude that $s^- \in S \cap [0, t^-]$ and $s^+ \in S \cap [t^+, 1]$.

Our goal now is to use a sequence of strictly increasing functions $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(S) \to \{0, 1\}$; then, from the known closeness properties of the class $\mathcal{S}$, we will be able to conclude that $\{0, 1\} \in \mathcal{S}$.

Since $\underline{s} \ge 0$ and $\overline{s} \le 1$, we conclude that $0 < t^- < s_0 < t^+ < 1$.

Let us select $\varepsilon > 0$ for which $0 < \varepsilon < s_0 < 1 - \varepsilon < 1$. For this property to hold, we must have $\varepsilon > 0$, $\varepsilon < s_0$, $s_0 < 1 - \varepsilon$, and $1 - \varepsilon < 1$. The last inequality is equivalent to the first inequality $\varepsilon > 0$. The third inequality, $s_0 < 1 - \varepsilon$, is equivalent to $\varepsilon < 1 - s_0$. The conditions $\varepsilon < s_0$ and $\varepsilon < 1 - s_0$ can be reformulated as $\varepsilon < \min(s_0, 1 - s_0)$. So, we are looking for the values ε which are strictly between 0 and $\min(s_0, 1-s_0)$, i.e., for the values ε from the open interval $(0, \min(s_0, 1-s_0))$.

For every ε from this interval $(0, \min(s_0, 1-s_0))$, we have $0 < \varepsilon < s_0$ and $\varepsilon < 1 - s_0$, i.e., equivalently, $s_0 < 1 - \varepsilon$. So, we have $0 < \varepsilon < s_0 < 1 - \varepsilon < 1$.

We now have two strictly increasing sequences of numbers: $0 < t^- < s_0 < t^+ < 1$ and $0 < \varepsilon < s_0 < 1 - \varepsilon < 1$. Let us us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ which maps the first sequence into the second one, i.e., for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(t^-) = \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, $\varphi_\varepsilon(t^+) = 1 - \varepsilon$, and $\varphi_\varepsilon(1) = 1$.

This piece-wise linear function takes the following form:

- for $0 \le x \le t^-$, we have
$$\varphi_\varepsilon(x) = \frac{\varepsilon}{t^-} \cdot x;$$
- for $t^- \le x \le s_0$, we have
$$\varphi_\varepsilon(x) = \varepsilon + \frac{s_0 - \varepsilon}{s_0 - t^-} \cdot (x - t^-);$$
- for $s_0 \le x \le t^+$, we have
$$\varphi_\varepsilon(x) = s_0 + \frac{(1-\varepsilon) - s_0}{t^+ - s_0} \cdot (x - s_0);$$
- finally, for $t^+ \le x \le 1$, we have
$$\varphi_\varepsilon(x) = (1-\varepsilon) + \frac{1 - (1-\varepsilon)}{1 - t^+} \cdot (x - t^+).$$

Since the class $\mathcal{S}$ is closed under arbitrary automorphisms, we conclude that the set $\varphi_\varepsilon(S)$ also belongs to the class $\mathcal{S}$.

For the value $s^- = \inf S$, we have shown that $0 \le s^- < t^-$. The function $\varphi_\varepsilon(x)$ is strictly increasing, from $0 \le s^- < t^-$, $\varphi_\varepsilon(0) = 0$, and $\varphi_\varepsilon(t^-) = \varepsilon$, we conclude that $0 \le \varphi_\varepsilon(s^-) < \varepsilon$. Since $s^- \in S$, we have $\varphi_\varepsilon(s^-) \in \varphi_\varepsilon(S)$. Thus, the set $\varphi_\varepsilon(S)$ contains a point from the interval $[0, \varepsilon]$.

Similarly, from $t^+ < s^+ \le 1$, $\varphi_\varepsilon(t^+) = 1-\varepsilon$, $\varphi_\varepsilon(1) = 1$, and $s^+ \in S$, we conclude that $1-\varepsilon < \varphi_\varepsilon(s^+) \le 1$, and thus that the set $\varphi_\varepsilon(S)$ contains a point $\varphi_\varepsilon(t^+)$ from the interval $[1-\varepsilon, 1]$.

Here, $\varphi_\varepsilon([0,t^-]) = [0,\varepsilon]$ and $\varphi_\varepsilon([t^+,1]) = [1-\varepsilon,1]$. So, from $S \subseteq [0,t^-] \cup [t^+,1]$, it follows that $\varphi_\varepsilon(S) \subseteq [0,\varepsilon] \cup [1-\varepsilon,1]$.

In the limit $\varepsilon \to 0$, we conclude that the sequence of sets $\varphi_\varepsilon(S)$ tends to the set $\{0,1\}$. Thus, the class $\mathcal{S}$ indeed contains the set $\{0,1\}$.

3.2°. We have now proved that the class $\mathcal{S}$ indeed contains the set $\{0,1\}$. Let us now prove that the class $\mathcal{S}$ contains a set $\{0,p\}$ for an arbitrary $p \in [0,1]$.

Indeed, we have proven that $\{0,1\} \in \mathcal{S}$. By definition, the class $\mathcal{S}$ contains all 1-element sets. In particular, it contains the set $\{p\}$. Also by definition, the class $\mathcal{S}$ is closed under $\wedge$. Thus, $\mathcal{S}$ must contain the set

$$\{0,1\} \wedge \{p\} \stackrel{\text{def}}{=} \{\min(a,b) : a \in \{0,1\}, b \in \{p\}\}.$$

By considering all possible elements of $\{0,1\}$ and of $\{p\}$, we conclude that

$$\{0,1\} \wedge \{p\} = \{\min(0,1), \min(p,1)\} = \{0,p\}.$$

Thus, we indeed have $\{0,p\} \in \mathcal{S}$.

3.3°. In the previous statement, we have proved that an arbitrary 2-element set containing 0 belongs to the class $\mathcal{S}$. Let us now prove that an arbitrary finite set containing 0 belongs to this class.

Indeed, if we sort all non-zero elements, we can represent an arbitrary finite set containing 0 as $\{0, p_1, p_2, \ldots, p_n\}$ with $0 < p_1 < p_2 < \ldots < p_n$. Let us prove, by induction over n, that an arbitrary finite set of this type indeed belongs to the class $\mathcal{S}$.

The base case of this induction is the case of $n = 1$, i.e., the statement that every set of the type $\{0, p_1\}$ belongs to the class $\mathcal{S}$. We already proven this in Part 3.2 of this proof.

Let us now consider the induction step. Let us assume that we have already proven this statement for n, and we want to prove it for $n+1$. In other words, we want to prove that an arbitrary set $\{0, p_1, \ldots, p_{n+1}\}$ with $p_1 < p_2 \ldots < p_{n+1}$ belongs to the class $\mathcal{S}$.

From the case of n, we conclude that the auxiliary set $\{0, p_2, \ldots, p_{n+1}\}$ (which contains n non-zero elements) belongs to the class $\mathcal{S}$. In Part 3.2 of this proof, we have also shown that an arbitrary set of the type $\{0, p\}$ belongs to the class $\mathcal{S}$ – including the set $\{0, p_1\}$.

Since the class $\mathcal{S}$ is closed under $\vee$, from $\{0, p_2, \ldots, p_{n+1}\} \in \mathcal{S}$ and $\{0, p_1\} \in \mathcal{S}$, we can now conclude that

$$\{0, p_2, \ldots, p_{n+1}\} \vee \{0, p_1\} \in \mathcal{S}.$$

By definition of $A \vee B$,

$$\{0, p_2, \ldots, p_{n+1}\} \vee \{0, p_1\} = \{\max(a, b) : a \in \{0, p_2, \ldots, p_{n+1}\}, b \in \{0, p_1\}\} =$$

$$\{\max(0,0), \max(p_2, 0), \ldots, \max(p_{n+1}, 0), \max(0, p_1), \max(p_2, p_1), \ldots, \max(p_{n+1}, p_1)\}.$$

Since $0 < p_1 < p_2 \ldots < p_{n+1}$, we conclude that $\max(p_i, 0) = \max(0, p_i) = p_i$ and $\max(p_i, p_1) = p_i$. Thus,

$$\{0, p_2, \ldots, p_{n+1}\} \vee \{0, p_1\} = \{0, p_1, p_2, \ldots, p_{n+1}\}.$$

So, we conclude that the set $\{0, p_1, p_2, \ldots, p_{n+1}\}$ indeed belongs to the class $\mathcal{S}$.

The induction step is proven, and so is the statement.

3.4°. We have just proved that an arbitrary finite set containing 0 belongs to the class $\mathcal{S}$. Let us now prove that an arbitrary finite set (not necessarily containing 0) also belongs to the class $\mathcal{S}$.

Indeed, if we sort all the elements of a finite set, we can represent it in the form $\{p_1, p_2, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_n$. We have already shown that $\{0, p_1, p_2, \ldots, p_n\} \in \mathcal{S}$. By definition of class $\mathcal{S}$, this class contains all 1-element sets. In particular, it contains the set $\{p_1\}$.

Since the class $\mathcal{S}$ is closed under $\vee$, from $\{0, p_1, p_2, \ldots, p_n\} \in \mathcal{S}$ and $\{p_1\} \in \mathcal{S}$, we can now conclude that

$$\{0, p_1, p_2, \ldots, p_n\} \vee \{p_1\} \in \mathcal{S}.$$

By definition of $A \vee B$,

$$\{0, p_1, p_2, \ldots, p_n\} \vee \{p_1\} = \{\max(a, b) : a \in \{0, p_1, p_2, \ldots, p_n\}, b \in \{p_1\}\} =$$

$$\{\max(0, p_1), \max(p_1, p_1), \max(p_2, p_1), \ldots, \max(p_n, p_1)\}.$$

Since $0 < p_1 < p_2 \ldots < p_n$, we conclude that $\max(0, p_1) = p_1$ and $\max(p_i, p_1) = p_i$ for all i. Thus,

$$\{0, p_1, p_2, \ldots, p_n\} \vee \{p_1\} = \{p_1, p_2, \ldots, p_n\}.$$

So, we conclude that the set $\{p_1, p_2, \ldots, p_n\}$ indeed belongs to the class $\mathcal{S}$. The statement is proven.

3.5°. We have proved that class S contains arbitrary finite sets. Now we will prove that the class $\mathcal{S}$ contains an arbitrary closed set $S \subseteq [0, 1]$.

Indeed, for every ε, we can consider a finite approximation S_ε to the set S, by taking the set of all the grid points $k \cdot \varepsilon$ (with integer k) for which $[k \cdot \varepsilon, (k+1) \cdot \varepsilon] \cap S \neq \emptyset$. One can easily check that in the limit $\varepsilon \to 0$, we have $S_\varepsilon \to S$. Thus, from the fact that the class $\mathcal{S}$ contains all finite sets S_ε, we conclude that the class $\mathcal{S}$ must also contain their limit S.

The theorem is proven.

3.2 Proof of Theorem 2

0°. The main difference between the situation studied Theorem 1 and the situation studied in Theorem 2, is that in Theorem 2 we are no longer requiring that the class $\mathcal{S}$ contains all 1-element sets. Instead, we require that this class $\mathcal{S}$ contains a set S which contains neither 0 not 1.

This condition is weaker than the condition that $\mathcal{S}$ contains all 1-element sets. Indeed, if $\mathcal{S}$ contains all 1-element sets, then, in particular, it contains a set $\{0.5\}$ which contains neither 0 nor 1.

We need to show that even under this weaker condition, the class $\mathcal{S}$ still coincides with one of the classes P, I, and C.

1°. Let us first prove that the class $\mathcal{S}$ contains a one-point set $\{s_0\}$ for some $s_0 \in (0,1)$, i.e., for some value s_0 for which $s_0 \neq 0$ and $s_0 \neq 1$.

Let us select one of the sets $S \in \mathcal{S}$ which does not contain 0 or 1. If this set is already a one-point set, then it has the form $S = \{s_0\}$ for some element s_0. Since we assumed that $0 \notin S$ and $1 \notin S$, we thus conclude that $s_0 \neq 0$ and $s_0 \neq 1$. So, in this case, the statement is proven.

It is therefore sufficient to only consider the case when the selected set S is not a one-point set.

From the proof of Theorem 1, we already know that for $s^- = \inf S$ and $s^+ = \sup S$, we have $s^- \in S$, $s^+ \in S$, and $S \subseteq [s^-, s^+]$. Since $0 \notin S$ and $1 \notin S$, we thus conclude that $s^- \neq 0$ (i.e., $s^- > 0$) and $s^+ \neq 1$ (i.e., $s^+ < 1$). So, $S \subseteq [s^-, s^+]$ for some s^- and s^+ for which $0 < s^- \leq s^+ < 1$.

In the degenerate case $s^- = s^+$, we would have a degenerate interval $[s^-, s^+]$ and the set S would be a one-point set. Since we assumed that S is not a one-point set, we have $s^- < s^+$. Let us denote the midpoint of the interval $[s^-, s^+]$ by $s_0 = \dfrac{\underline{s} + \overline{s}}{2}$. Here, $0 < s_0 < 1$.

For every $\varepsilon \in (0, \min(s_0, 1 - s_0))$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(s^-) = s_0 - \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, $\varphi_\varepsilon(s^+) = s_0 + \varepsilon$, and $\varphi_\varepsilon(1) = 1$.

Thus, our piece-wise linear function $\varphi_\varepsilon(x)$ has the following form:

- for $0 \leq x \leq s^-$, we have

$$\varphi_\varepsilon(x) = \frac{s_0 - \varepsilon}{s^-} \cdot x;$$

- for $s^- \leq x \leq s_0$, we have

$$\varphi_\varepsilon(x) = s_0 - \varepsilon + \frac{s_0 - (s_0 - \varepsilon)}{s_0 - s^-} \cdot (x - s^-);$$

- for $s_0 \le x \le s^+$, we have

$$\varphi_\varepsilon(x) = s_0 + \frac{(s_0 + \varepsilon) - s_0}{s^+ - s_0} \cdot (x - s_0);$$

- finally, for $s^+ \le x \le 1$, we have

$$\varphi_\varepsilon(x) = s_0 + \varepsilon + \frac{1 - (s_0 + \varepsilon)}{1 - s^+} \cdot (x - s^+).$$

Then, from the fact that $S \in \mathcal{S}$ and $S \subseteq [s^-, s^+]$, and from the requirement that a set-valued extension of fuzzy logic must be closed under arbitrary automorphism of $([0,1], \wedge, \vee)$, we conclude that $\varphi_\varepsilon(S) \in \mathcal{S}$, $\varphi_\varepsilon(S) \subseteq \varphi([s^-, s^+]) = [s_0 - \varepsilon, s_0 + \varepsilon]$. In the limit $\varepsilon \to 0$, the sets $[s_0 - \varepsilon, s_0 + \varepsilon]$ tend to a one-point set $\{s_0\}$. Thus, the class $\mathcal{S}$ indeed contains a one-point set $\{s_0\}$ which is different from $\{0\}$ and from $\{1\}$.

2°. Now that we proved that class $\mathcal{S}$ contains at least one 1-element set $\{s_0\}$, let us show that $\mathcal{S}$ in fact contains all one-element sets $\{s\}$ for which $s \in (0, 1)$.

Indeed, for every $s \in (0, 1)$, we can construct a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s_0) = s$, and $\varphi(1) = 1$.

- for $0 \le x \le s_0$, we have

$$\varphi(x) = \frac{s}{s_0} \cdot x;$$

- for $s_0 \le x \le 1$, we have

$$\varphi(x) = s + \frac{1 - s}{1 - s_0} \cdot (x - s_0).$$

For this function $\varphi(x)$, we have $\varphi(\{s_0\}) = \{s\}$, so indeed $\{s\} \in \mathcal{S}$. This is true since we have defined before that invariance under automorphism is a property of S.

3°. The last thing to prove, is that the class $\mathcal{S}$ contains all one-element sets, not just those that are not different from $\{0\}$ or $\{1\}$.

The only missing one-point sets are the sets $\{0\}$ and $\{1\}$. The first set ($\{0\}$) can be represented as a limit of sets $\{1/n\} \in \mathcal{S}$: as $n \to \infty$, in the limit, we have $\{1/n\} \to \{0\}$. So, due to the closeness of the class $\mathcal{S}$, the set $\{0\}$ also belongs to the class $\mathcal{S}$.

The second set $\{1\}$, in its turn, is the limit of sets $\{1 - 1/n\} \in \mathcal{S}$. As $n \to \infty$, we have $1/n \to 0$ and $\{1 - 1/n\} \to \{1 - 0\} \to \{1\}$. Thus we also have $\{1\} \in \mathcal{S}$.

Thus, the condition (i) is satisfied, and the result follows from Theorem 1.

3.3 Proof of Theorem 3

1°. One can easily check that an arbitrary union of the above classes indeed satisfies conditions (i′) and (ii)-(iv).

2°. Let us first consider the case when there exists a set $S \in \mathcal{S}$ which contains both 0 and 1 but is different from the interval $[0, 1]$.

Since the set $S \in \mathcal{S}$ is different from $[0, 1]$ and contain 0 and 1, it must have "holes", i.e., it must have a non-empty complement $[0, 1] - S$.

Example. A simple example of a set which contains both 0 and 1 but which is different from the entire interval $[0, 1]$ is the 2-element set $S = \{0, 1\}$. This set S is different than the interval $[0, 1]$ since it does not contain any points s_0 such that $s_0 \in (0, 1)$: e.g., it does not contain the point $s_0 = 0.5$. For this set, the complement $[0, 1] - s$ is the entire open interval $(0, 1)$.

2.1°. Let us first consider the case when every set $S \in \mathcal{S}$ which contains both 0 and 1 and which is different from the entire interval $[0, 1]$ has only one hole, i.e., its complement is a connected interval.

By definition, all such sets have the form $[0, a] \cup [b, 1]$ with $a < b$.

$2.1.1^\circ$. Let us first consider the situation when the class $\mathcal{S}$ contains a set $[0, a] \cup [b, 1]$ with $a > 0$ and $b < 1$.

Let us show that by applying an appropriate $\varphi(x)$, we can show that with of every other set $[0, a'] \cup [b', 1]$ of this type – i.e., for which $a' > 0$ and $b' < 1$ – also belongs to $\mathcal{S}$. Indeed, we can define the following strictly increasing piece-wise linear function:

- for $0 \le x \le a$, we have
$$\varphi(x) = \frac{a'}{a} \cdot x;$$
- for $a \le x \le b$, we have
$$\varphi(x) = a' + \frac{b' - a'}{b - a} \cdot (x - a);$$
- finally, for $b \le x \le 1$, we have
$$\varphi(x) = b' + \frac{1 - b'}{1 - b} \cdot (x - b).$$

For this function, $\varphi([0,a] \cup [b,1]) = [0,a'] \cup [b',1]$. Since $[0,a] \cup [b,1] \in \mathcal{S}$, we thus conclude that $\varphi([0,a] \cup [b,1]) = [0,a'] \cup [b',1] \in \mathcal{S}$.

We want to prove that in this case, the class $\mathcal{S}$ coincides with I_{01}. To complete the proof, we must show that the class $\mathcal{S}$ contains the degenerate unions $\{0\} \cup [\overline{s}, 1]$ with $\overline{s} < 1$, $[0, \underline{s}] \cup \{1\}$ with $\underline{s} > 0$, $\{0\} \cup \{1\}$, and $[0,1]$.

To prove that the union $\{0\} \cup [\overline{s}, 1]$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, 1/n] \cup [\overline{s}, 1]$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1/n \to 0$, $[0, 1/n] \to \{0\}$ and thus, $[0, 1/n] \cup [\overline{s}, 1] \to \{0\} \cup [\overline{s}, 1]$. Since the class $\mathcal{S}$ is closed, we conclude that $\{0\} \cup [\overline{s}, 1] \in \mathcal{S}$.

Similarly, to prove that the union $[0, \underline{s}] \cup \{1\}$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, \underline{s}] \cup [1 - 1/n, 1]$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1 - 1/n \to 1$, $[1 - 1/n, 1] \to \{1\}$ and thus, $[0, \underline{s}] \cup [1 - 1/n, 1] \to [0, \underline{s}] \cup \{1\}$. Since the class $\mathcal{S}$ is closed, we conclude that $[0, \underline{s}] \cup \{1\} \in \mathcal{S}$.

To prove that the union $\{0\} \cup \{1\}$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, 1/n] \cup \{1\}$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1/n \to 0$, $[0, 1/n] \to \{0\}$ and thus, $[0, 1/n] \cup \{1\} \to \{0\} \cup \{1\}$. Since the class $\mathcal{S}$ is closed, we conclude that $\{0\} \cup \{1\} \in \mathcal{S}$.

To prove that the interval $[0,1]$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, 0.5 - 1/n] \cup [0.5 + 1/n, 1]$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1/n \to 0$, $[0, 0.5 - 1/n] \to [0, 0.5]$, $[0.5 + 1/n, 1] \to [0.5, 1]$, and thus,

$$[0, 0.5 - 1/n] \cup [0.5 + 1/n, 1] \to [0, 0.5] \cup [0.5, 1] = [0, 1].$$

Since the class $\mathcal{S}$ is closed, we conclude that $[0, 1] \in \mathcal{S}$.

The statement is proven.

2.1.2°. To complete the proof for the case 2.1, it is therefore sufficient to consider the remaining situation when for every set $[0, a] \cup [b, 1] \in \mathcal{S}$, we have either $a = 0$ or $b = 1$. In other words, every class from the class $\mathcal{S}$ has the form $\{0\} \cup [b, 1]$ or the form $[0, a] \cup \{1\}$.

2.1.3°. Let us first consider the case when the class $\mathcal{S}$ contains a set $\{0\} \cup [b, 1]$ with $b < 1$.

Let us show that by applying an appropriate $\varphi(x)$, we can show that with of every other set $\{0\} \cup [b', 1]$ of this type – i.e., for which $b' < 1$ – also belongs to $\mathcal{S}$. Indeed, we can define the following strictly increasing piece-wise linear function:

- for $0 \le x \le b$, we have
$$\varphi(x) = \frac{b'}{b} \cdot x;$$
- for $b \le x \le 1$, we have
$$\varphi(x) = b' + \frac{1 - b'}{1 - b} \cdot (x - b).$$

For this function, $\varphi(\{0\} \cup [b, 1]) = \{0\} \cup [b', 1]$. Since $\{0\} \cup [b, 1] \in \mathcal{S}$, we thus conclude that $\varphi(\{0\} \cup [b, 1]) = \{0\} \cup [b', 1] \in \mathcal{S}$.

We want to prove that in this case, the class $\mathcal{S}$ coincides with I_1^+. To complete the proof, we must show that the class $\mathcal{S}$ contains the degenerate unions $\{0\} \cup \{1\}$ and $[0, 1]$.

To prove that the union $\{0\} \cup \{1\}$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $\{0\} \cup [1 - 1/n, 1]$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1 - 1/n \to 1$, $[1 - 1/n, 1] \to \{1\}$ and thus, $\{0\} \cup [1 - 1/n, 1] \to \{0\} \cup \{1\}$. Since the class $\mathcal{S}$ is closed, we conclude that $\{0\} \cup \{1\} \in \mathcal{S}$.

To prove that the interval $[0, 1]$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $\{0\} \cup [1/n, 1]$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1/n \to 0$, $[1/n, 1] \to [0, 1]$, and thus, $\{0\} \cup [1/n, 1] \to \{0\} \cup [0, 1] = [0, 1]$. Since the class $\mathcal{S}$ is closed, we conclude that $[0, 1] \in \mathcal{S}$.

The statement is proven.

2.1.4°. Let us now consider the case when the class $\mathcal{S}$ contains a set $[0, a] \cup \{1\}$ with $a > 0$.

Let us show that by applying an appropriate $\varphi(x)$, we can show that with of every other set $[0, a'] \cup \{1\}$ of this type – i.e., for which $a' > 0$ – also belongs to $\mathcal{S}$. Indeed, we can define the following strictly increasing piece-wise linear function:

- for $0 \le x \le a$, we have
$$\varphi(x) = \frac{a'}{a} \cdot x;$$
- for $a \le x \le 1$, we have
$$\varphi(x) = a' + \frac{1 - a'}{1 - a} \cdot (x - a).$$

For this function, $\varphi([0, a] \cup \{1\}) = [0, a'] \cup \{1\}$. Since $[0, a] \cup \{1\} \in \mathcal{S}$, we thus conclude that $\varphi([0, a] \cup \{1\}) = [0, a] \cup \{1\} \in \mathcal{S}$.

We want to prove that in this case, the class $\mathcal{S}$ coincides with I_0^+. To complete the proof, we must show that the class $\mathcal{S}$ contains the degenerate unions $\{0\} \cup \{1\}$ and $[0, 1]$.

To prove that the union $\{0\} \cup \{1\}$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, 1/n] \cup \{1\}$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1/n \to 0$, $[0, 1/n] \to \{0\}$ and thus, $[0, 1/n] \cup \{0\} \to \{0\} \cup \{1\}$. Since the class $\mathcal{S}$ is closed, we conclude that $\{0\} \cup \{1\} \in \mathcal{S}$.

To prove that the interval $[0, 1]$ belongs to the class $\mathcal{S}$, we can use the already proven fact that for every n, the union $[0, 1 - 1/n] \cup \{1\}$ belongs to $\mathcal{S}$. When $n \to \infty$, we have $1 - 1/n \to 1$, $[0, 1/n - 1] \to [0, 1]$, and thus, $[0, 1/n - 1] \cup \{1\} \to [0, 1] \cup \{1\} = [0, 1]$. Since the class $\mathcal{S}$ is closed, we conclude that $[0, 1] \in \mathcal{S}$.

The statement is proven.

2.1.5°. The only remaining subcase is when the class $\mathcal{S}$ consists of only one set $\{0\} \cup \{1\}$.

2.2°. The only remaining situation is when there is a set $S \in \mathcal{S}$ that contains 0, 1, and at least two holes. This is the case when $\underline{s} < \overline{s}$. In this case we have $S \subseteq [0, s_1] \cup [s_2, s_3] \cup [s_4, 1]$ for some values $0 \le s_1 < s_2 \le s_2 < s_4 \le 1$ for which $0 \in S$, $S \cup [s_2, s_3] \ne \emptyset$, and $1 \in S$.

In this case, by using appropriate functions $\varphi_\varepsilon(x)$ and tending to the limit $\varepsilon \to 0$, we "compress" the interval $[0, s_1]$ into a single point 0, the interval $[s_4, 1]$ into a single point 1, and the interval $[s_2, s_3]$ into a single midpoint $s_0 \in (0, 1)$. Thus, we conclude that a 3-point set $S_0 \stackrel{\text{def}}{=} \{0, s_0, 1\}$ belongs to the class $\mathcal{S}$.

For an arbitrary value $s \in (0, 1)$, by using a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s_0) = s$, and $\varphi(1) = 1$, we can now conclude that $\varphi(S_0) = \{0, s, 1\} \in \mathcal{S}$.

As such a function $\varphi(x)$, we can take, e.g., the following strictly increasing piece-wise linear function:

- for $0 \le x \le s_0$, we have
$$\varphi(x) = \frac{s}{s_0} \cdot x;$$
- for $s_0 \le x \le 1$, we have
$$\varphi(x) = s + \frac{1 - s}{1 - s_0} \cdot (x - s_0).$$

For $s = 0$ and $s = 1$, we can take a limit and thus conclude that $\{0, s, 1\} \in \mathcal{S}$ for all values $s \in [0, 1]$.

It is easy to check that for every two sets A and A',

$$(\{0, 1\} \cup A) \vee (\{0, 1\} \cup A') = \{0, 1\} \cup (A \cup A'). \tag{3.1}$$

Indeed, by definition of the set "or" operation, every element of $S \vee S'$ has the form $s \vee s' = \max(s, s')$ for some $s \in S$ and $s' \in S'$ and is, thus, equal either to $s \in S$ or to $s' \in S'$. Thus, every element of the set $S \vee S'$ belongs to the union $S \cup S'$. On the other hand, every element $s \in S$ can be represented as $s \vee 0$, and every element $s' \in S'$ as $0 \vee s'$ – hence every element of the union indeed belongs to $A \vee A'$.

We start with sets $\{0, s, 1\}$ which correspond to 1-element sets $A = \{s\}$. An arbitrary finite set can be represented as a union of its one-element subsets. Thus, due to the equality (3.1), we can conclude that $\mathcal{S}$ contains sets $\{0, 1\} \cup A$ for an arbitrary finite A – i.e., that $\mathcal{S}$ contains an arbitrary finite set which contains 0 and 1.

Since every closed set can be represented as a limit of finite sets, in the limit, we conclude that $\mathcal{S}$ contains an arbitrary closed set which contains 0 and 1, i.e., $C_{01} \subseteq \mathcal{S}$.

3°. If we have a set $S_0 \neq [0, 1]$ that contains 0, does not contain 1, and is different from $\{0\}$, then we also have two possibilities:

- either all such sets are intervals,
- or one of them is not an interval.

3.1°. Let us first consider a situation in which all such sets $S \in \mathcal{S}$ are intervals. Since $0 \in S$, they can only be intervals of type $[0, s]$. In particular, the interval S_0 (whose existence we have just assumed) is different from $\{0\}$ and does not contain 1, so it has the form $[0, s_0]$ for some $s_0 \in (0, 1)$.

By using an appropriate $\varphi(x)$, we conclude that every interval of the type $[0, s]$ with $s \in (0, 1)$ also belongs to $\mathcal{S}$. By taking a limit, we deduce that $\mathcal{S}$ contains all intervals $[0, s]$, i.e., that $I_0 \subseteq \mathcal{S}$.

3.2°. Let us now consider a situation in which there exist a non-interval set $S \in \mathcal{S}$ which contains 0 but does not contain 1.

As we have shown earlier, $S \subseteq [s^-, s^+]$, with $s^- = \inf S \in S$ and $s^+ = \sup S \in S$. Since $1 \notin S$, we have $s^+ < 1$; since $S \neq \{0\}$, we have $0 < s^+$. Due to the fact the set S is not an interval, it must have a hole, i.e., we have $S \subseteq [0, s_1] \cup [s_2, s^+]$ for some valued $0 \leq s_1 < s_2 \leq s^+ < 1$ for which $0 \in S$ and $s^+ \in S$.

By using appropriate functions $\varphi_\varepsilon(x)$, we "compress" the interval $[0, s_1]$ into a single point 0 and the interval $[s_2, s^+]$ into a single point $s^+ \in (0, 1)$. Thus, we conclude that a 2-point set $S_0 \stackrel{\text{def}}{=} \{0, s^+\}$ belongs to the class $\mathcal{S}$.

For an arbitrary value $s \in (0, 1)$, by using a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s^+) = s$, and $\varphi(1) = 1$, we can now conclude that $\varphi(S_0) = \{0, s\} \in \mathcal{S}$.

For $s = 0$ and $s = 1$, we can take a limit and thus conclude that $\{0, s\} \in \mathcal{S}$ for all values $s \in [0, 1]$.

It is easy to check that for every two sets A and A',

$$(\{0\} \cup A) \vee (\{0\} \cup A') = \{0\} \cup (A \cup A'). \tag{3.2}$$

We start with sets $\{0, s\}$ which correspond to 1-element sets $A = \{s\}$. An arbitrary finite set can be represented as a union of its one-element subsets. Thus, due to the equality (3.2), we can conclude that $\mathcal{S}$ contains sets $\{0\} \cup A$ for an arbitrary finite A – i.e., that $\mathcal{S}$ contains an arbitrary finite set which contains 0. In the limit, we conclude that $\mathcal{S}$ contains an arbitrary closed set which contains 0, i.e., that $C_0 \subseteq \mathcal{S}$.

4°. If we have a set $S_0 \neq [0, 1]$ that contains 1, does not contain 0, and is different from $\{1\}$, then we can use a similar argument to conclude that either $I_1 \subseteq \mathcal{S}$ or $C_1 \subseteq \mathcal{S}$. The only difference is that instead of (3.2), we must use a dual formula

$$(\{1\} \cup A) \wedge (\{1\} \cup A') = \{1\} \cup (A \cup A'). \tag{3.3}$$

The theorem is proven.

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Curriculum Vitae

Gilbert Ornelas Duarte was born in El Paso, Texas on May 20, 1982. The proud son of Gilberto and Maria Teresa Ornelas, Gilbert lived in Ciudad Juarez, Chihuahua for the first eleven years of his life before coming back to El Paso on July 1993 ready to start the 6th grade. In the year 2000, he graduated from Mt. View High School and enrolled at The University of Texas at El Paso in the fall of that year. Wanting to know what computer science was about, Gilbert enrolled in the introductory course to computer science in the spring of 2001. Two years later, he became a peer leader for that same course and later filled in as a teaching assistant for a data structures course. During his time as a computer science undergraduate, he had the opportunity to participate in internships with the Arctic Region Supercomputing Center (2003) and the National Aeronautics and Space Agency (2004). In the fall of 2003, Gilbert became a research assistant working for the Pan American Center for Earth and Environmental Studies under Dr. Ann Gates. Upon graduating in May 2005, Gilbert enrolled in the Master's program at UTEP and received the Louis Stokes Alliance for Minority Participation fellowship. During his time at UTEP, Gilbert has also worked as a software developer for Electro Systems Engineers, Inc., and as software analyst for XIMIS, Inc. where he has worked since September 2006.

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