Why Some Power Laws Are Possible And Some Are Not

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1. Power Laws Are Ubiquitous

- In many application areas, the dependence between two quantities \( x \) and \( y \) is described by the formula

\[
y = A \cdot x^a \text{ for some } a \text{ and } A.
\]

- Such dependencies are known as *power laws*.
- Power laws are truly ubiquitous.
- They describe how the aerodynamic resistance force depends on the plane’s velocity.
- They describe how the perceived signal depends on the intensity of the signal that we hear and see.
- They describe how the mass of celestial structures depends on the structure’s radius, etc.
2. Sometimes, Not All Power Laws Are Possible

- The parameters $A$ and $a$ have to be determined from the experiment.
- In some application areas, all pairs $(A, a)$ are possible.
- In some other applications areas, however, not all such pairs are possible.
- Sometimes, $a$ is fixed, and $A$ can take all possible values.
- In other application areas:
  - we have different values of $A$,
  - but for each $A$, we can only have one specific value of $a$. 
3. Not All Power Laws Are Possible (cont-d)

- One such example can be found in transportation engineering.

- It describes the dependence of number $y$ of cycles until fatigue failure on the initial strain $x$.

- In many such situations, the value of $a$ corresponding to $A$ is determined by the following empirical formula

  $$a = c_0 + c_1 \cdot \ln(A).$$

- The case when the value $a$ is fixed can be viewed as a particular case $c_1 = 0$ of this empirical formula.
4. Resulting Challenge

- How can we explain the formula \( a = c_0 + c_1 \cdot \ln(A) \)?

- In this talk, we provide a theoretical explanation for this formula.

- To come up with this explanation:
  - we recall the reason why power laws are ubiquitous in the first place
  - because they correspond to scale-invariant dependencies.

- We then use the scale-invariance idea to explain the ubiquity of the desired formula.
5. Power Laws and Scale Invariance

- The main purpose of data processing is to deal with physical quantities.
- However, in practice, we only deal with the numerical values of these quantities.
- What is the difference?
- The difference is that:
  - to get a numerical value,
  - we need to select a measuring unit for measuring the quantity.
- If:
  - we replace the original measuring unit with a new one which is \( \lambda \) times smaller,
  - then all numerical values are multiplied by \( \lambda \):

\[
x \rightarrow X = \lambda \cdot x.
\]
6. Power Laws and Scale Invariance (cont-d)

- For example, if we move from meters to centimeters:
  - all the numerical values will be re-scaled: multiplied by 100;
  - e.g., 1.7 m becomes $1.7 \cdot 100 = 170$ cm.
7. Scale-Invariance

- In many application areas, there is no fixed measuring unit.
- The choice of the measuring unit is rather arbitrary.
- In such situations, it is reasonable to require that:
  - the dependence \( y = f(x) \) between the quantities \( x \) and \( y \)
  - should not depend on the choice of the unit.
- Of course, this does not mean that \( y = f(x) \) imply \( y = f(X) = f(\lambda \cdot x) \) for the exact same function \( f(x) \).
- That would mean that \( f(\lambda \cdot x) = f(x) \) for all \( x \) and \( \lambda \).
- So \( f(x) \) is a constant and thus, that there is no dependence.
8. Scale-Invariance (cont-d)

- What we need to do to keep the same dependence is:
  - to accordingly re-scale $y$,
  - to $Y = \mu \cdot y$ for some $\mu$ depending on $\lambda$.

- For example, the area $y$ of a square is equal to the square of its size $y = x^2$.

- This formula is true if we use meters to measure length and square meters to measure area.

- The same formula holds if we use centimeters instead of meters.

- However, then, we should use square centimeters instead of square meters.

- In this case, $\lambda = 100$ corresponds to $\mu = 10000$. 
9. Scale-Invariance (cont-d)

- So, we arrive at the following definition of scale-invariance:
  
  - for every \( \lambda > 0 \) there exists a value \( \mu > 0 \) for which, for every \( x \) and \( y \),
  
  - the relation \( y = f(x) \) implies that \( Y = f(X) \) for \( X = \lambda \cdot x \) and \( Y = \mu \cdot y \).
10. Scale-Invariance and Power Laws

• It is easy to check that every power law is scale-invariant.
• Indeed, it is sufficient to take $\mu = \lambda^a$.
• Then, from $y = A \cdot x^a$ we get
  
  \[ Y = \mu \cdot y = \lambda^a \cdot y = \lambda^a \cdot A \cdot x^a = a \cdot (\lambda \cdot x)^a = a \cdot X^a. \]
• So, indeed $Y = f(X)$.
• It turns out that, vice versa, the only continuous scale-invariance dependencies are power laws.
• For differentiable functions $f(x)$, this can be easily proven.
• Indeed, by definition, scale-invariance means that $\mu(\lambda) \cdot f(x) = f(\lambda \cdot x)$.
• Since $f(x)$ is differentiable, $\mu(\lambda) = \frac{f(\lambda \cdot x)}{f(x)}$ is also differentiable, as the ratio of two differentiable functions.
11. Scale-Invariance and Power Laws (cont-d)

• Since \( f(x) \) and \( \mu(\lambda) \) are differentiable, we can differentiate the equality \( \mu(\lambda) \cdot f(x) = f(\lambda \cdot x) \) w.r.t. \( \lambda \):

\[
\mu'(\lambda) \cdot f(x) = x \cdot f'(\lambda \cdot x).
\]

• In particular, for \( \lambda = 1 \), we get \( \mu_0 \cdot f(x) = x \cdot f'(x) \), where \( \mu_0 \) \( \text{def} \) \( = \mu'(1) \), so \( \mu_0 \cdot f = x \cdot \frac{df}{dx} \).

• We can separate the \( x \) and \( f \) is we divide both sides by \( x \cdot f \) and multiply by \( dx \): \( \frac{df}{f} = \mu_0 \cdot \frac{dx}{x} \).

• Integrating both sides, we get \( \ln(f) = \mu_0 \cdot \ln(x) + c \), where \( c \) is the integration constant.

• Thus, for \( f = \exp(\ln(f)) \), we get

\[
f(x) = \exp(\mu_0 \cdot \ln(x) + c) = A \cdot x^a.
\]

• Here \( A \) \( \text{def} \) \( = \exp(c) \) and \( a \) \( \text{def} \) \( = \mu_0 \).
12. Main Idea

- In principle, for the corresponding application areas, we can have different values $A$ and $a$.
- This means that the value of the quantity $y$ is not uniquely determined by the value of the quantity $x$.
- There must be some other quantity $z$ that influences $y$: $y = F(x, z)$.
- Different situations – i.e., different pairs $(A, a)$ – are characterized by different values of the quantity $z$. 
13. Main Assumption

- For each fixed $z$, the dependence of $y$ on $x$ is described by a power law.

- Thus, when the value of $z$ is fixed, the dependence of $y$ on $x$ is scale-invariant.

- It is therefore reasonable to conclude that, vice versa:
  - for each fixed value $x$,
  - the dependence of $y$ on $z$ is also scale-invariant.
14. This Assumption Leads to the Desired Explanation

- Let us show that this assumption indeed explains the desired formula.
- For each $z$, the dependence of $y$ on $x$ is a power law:
  \[ F(x, z) = A(z) \cdot x^{a(z)}. \]
- Similarly, for each $x$, the dependence of $y$ on $z$ is also a power law:
  \[ F(x, z) = B(x) \cdot z^{b(x)}. \]
- Thus, $A(z) \cdot x^{a(z)} = B(x) \cdot z^{b(x)}$ for all $x$ and $z$. 
15. Explanation (cont-d)

- In particular, for $x = 1$, we get $A(z) = B(1) \cdot z^{b(1)}$.
- Similarly, for $z = 1$, we get $B(x) = A(1) \cdot x^{a(1)}$.
- Substituting these expressions into the above equality, we get $B(1) \cdot z^{b(1)} \cdot x^{a(z)} = A(1) \cdot x^{a(1)} \cdot z^{b(x)}$.
- In particular, for $x = e$, we get $B(1) \cdot z^{b(1)} \cdot e^{a(z)} = A(1) \cdot e^{a(1)} \cdot z^{b(e)}$.
- So, $\exp(a(z)) = \frac{A(1) \cdot \exp(a(1))}{B(1)} \cdot z^{b(e) - b(1)}$.
- From $A(z) = B(1) \cdot z^{b(1)}$, we conclude that $z^{b(1)} = \frac{A}{B(1)}$.
- Thus, $z = \frac{A^{1/b(1)}}{B(1)^{1/b(1)}}$. 
16. Explanation (cont-d)

- *Reminder:* 
  \[ z = \frac{A^{1/b(1)}}{B(1)^{1/b(1)}}. \]

- Substituting this expression for \( z \) into the formula for \( \exp(a) \), we get:
  \[ \exp(a) = \frac{A(1) \cdot \exp(a(1))}{B(1)} \cdot \frac{1}{B(1)^{(b(e)-b(1))/b(1)}} \cdot A^{(b(e)-b(1))/b(1)}}. \]

- So, \( \exp(a) = C_0 \cdot A^{c_1} \) for some values \( C_0 \) and \( c_1 \).

- Taking logarithms of both sides, we now get the desired dependence \( a = c_0 + c_1 \cdot \ln(A) \), where \( c_0 \overset{\text{def}}{=} \ln(C_0) \).

- So, we indeed have the desired derivation.
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