How to Generate “Nice” Cubic Polynomials – with Rational Coefficients, Rational Zeros and Rational Extrema: A Fast Algorithm

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1. Need for Nice Calculus-Related Examples

- After students learn the basics of calculus, they practice them graphing functions $y = f(x)$.
- They find the roots (zeros), i.e., values where $f(x) = 0$.
- They find the extreme points, i.e., values where the derivative $f'(x)$ is equal to 0.
- They find out whether $f(x)$ increases or decreases between extreme points – by checking the sign of $f'(x)$.
- They use this information – plus the values of $f(x)$ at several points $x$ – to graph the function.
- For this practice, students need examples for which they can compute both the zeros and the extreme points.
2. Cubic Polynomials: the Simplest Case When Such an Analysis Makes Sense

- The simplest possible functions are polynomials.
- For linear functions, the derivative is constant, so there are no extreme point.
- For quadratic functions, there is an extreme point.
- However, after studying quadratic equations, students already know how to graph the corresponding function.
- So, for quadratic polynomials, there is no need to use calculus.
- The simplest case when calculus tools are needed is the case of cubic polynomials.
3. To Make It Simpler For Students, It Is Desirable to Limit Ourselves to Rational Roots

- Students are much more comfortable with rational numbers than with irrational ones.
- Thus, it is desirable to have examples when all the coefficients, zeros, and extreme points of a are rational.
- Good news is that when we know that the roots are rational, it is (relatively) easy to find these roots.
- Indeed, for each rational root \( x = \frac{p}{q} \) of a polynomial \( a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_0 \) with integer coefficients:
  - the numerator \( p \) is a factor of \( a_0 \), and
  - the denominator \( q \) is a factor of \( a_n \).
- How can we find polynomials for which both zeros and extreme points are rational?
4. What Is Known and What We Do

• An algorithm for generating such polynomials was recently proposed.

• This algorithm, however, is not the most efficient one.

• For each tuple of the corresponding parameter values, it uses exhaustive trial-and-error search.

• In this presentation, we produce an efficient algorithm for producing nice polynomials.

• Namely, we propose simple computational formulas:
  – for each tuple of the corresponding parameters, these formulas produce a “nice” cubic polynomial;
  – every “nice” cubic polynomial can be thus generated.

• For each tuple, our algorithm requires the same constant number of elementary steps.
5. Analysis of the Problem

- A general cubic polynomial with rational coefficients has the form \( a \cdot X^3 + b \cdot X^2 + c \cdot X + d \).

- Roots and extreme points of \( f(x) \) do not change if we simply divide all its values by the same constant \( a \).

- Thus, it is sufficient to consider polynomials with only three parameters: \( X^3 + p \cdot X^2 + q \cdot X + r \), where

\[
p = \frac{b}{a}, \quad q = \frac{c}{a}, \quad r = \frac{d}{a}.
\]

- We can further simplify the problem if we replace \( X \) with \( x = X + \frac{p}{3} \), then we get \( x^3 + \alpha \cdot x + \beta \), where

\[
\alpha = q - \frac{p^2}{3} \quad \text{and} \quad \beta = r - \frac{p \cdot q}{3} + \frac{2p^3}{27}.
\]
6. **Analysis of the Problem (cont-d)**

- Let $r_1$, $r_2$, and $r_3$ denote rational roots of $x^3 + \alpha \cdot x + \beta$, then, we have
  \[ x^3 + \alpha \cdot x + \beta = (x - r_1) \cdot (x - r_2) \cdot (x - r_3). \]

- So, $r_1 + r_2 + r_3 = 0$, $\alpha = r_1 \cdot r_2 + r_2 \cdot r_3 + r_1 \cdot r_3$, and $\beta = -r_1 \cdot r_2 \cdot r_3$.

- Substituting $r_3 = -(r_1 + r_2)$ into these formulas, we get
  \[ \alpha = -(r_1^2 + r_1 \cdot r_2 + r_2^2) \] and $\beta = r_1 \cdot r_2 \cdot (r_1 + r_2)$. 

7. Using the Fact That the Extreme Points $x_0$ Should Also Be Rational

• Differentiating and equating the derivative to 0, we get

$$3x_0^2 - (r_1^2 + r_1 \cdot r_2 + r_2^2) = 0.$$ 

• This is equivalent to $3x_0^2 - 3y^2 - z^2 = 0$, where

$$y \overset{\text{def}}{=} \frac{r_1 + r_2}{2} \quad \text{and} \quad z \overset{\text{def}}{=} \frac{r_1 - r_2}{2}.$$ 

• If we divide both sides of this equation by $y^2$, we get

$$3X_0^2 - 3 - Z^2 = 0,$$ 

where $X_0 \overset{\text{def}}{=} \frac{x_0}{y}$ and $Z \overset{\text{def}}{=} \frac{z}{y}$.

• One of the solution of above equation is easy to find: namely, when $X_0 = -1$, we get $Z^2 = 0$ and $Z = 0$.

• This means that for every $y$, $x_0 = -y$, $y$ and $z = 0$ solve the above equation.
8. Using the Fact That the Extreme Points $x_0$ Should Also Be Rational (cont-d)

- We can now reconstruct $r_1$ and $r_2$ from $y$ and $z$ as $r_1 = y + z$ and $r_2 = y - z$.

- In our case, $r_1 = r_2 = y$, so $\alpha = -3y^2$ and $\beta = 2y^3$.

- We can then:
  - shift by a rational number $s$, $(x \to X = x + s)$, and
  - multiply all the coefficients by an arbitrary rational number $a$.

- Then, we get
  $$b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3).$$
9. Using the General Algorithm for Finding All Rational Solutions to a Quadratic Equation

- We have already found a solution of the equation $3X_0^2 - 3 - Z^2 = 0$, corresponding to $X_0 = -1$: then $Z = 0$.

- Every other solution $(X_0, Z)$ can be connected to this simple solution $(-1, 0)$ by a straight line.

- A general equation of a straight line passing through the point $(-1, 0)$ is $Z = t \cdot (X_0 + 1)$.

- When $X_0$ and $Z$ are rational, $t = \frac{Z}{X_0 + 1}$ is rational.

- Substituting this expression for $Z$ into the equation, we get $3X_0^2 - 3 - t^2 \cdot (X_0 + 1)^2 = 0$.

- Since $X_0 \neq -1$, we can divide both sides by $X_0 + 1$. then $3 \cdot (X_0 - 1) - t^2 \cdot (X_0 + 1) = 0$, hence

$$X_0 = \frac{3 + t^2}{3 - t^2} \text{ and } Z = \frac{6t}{3 - t^2}.$$
10. Towards a General Description of All “Nice” Polynomials

- For every rational \( y \), we can now take \( x_0 = y \cdot X_0 \), \( y \), and \( z = y \cdot Z = \frac{6yt}{3 - t^2} \).

- Based on \( y \) and \( z \), we can compute \( r_1 = y + z \) and \( r_2 = y - z \).

- Then, we can compute \( \alpha \) and \( \beta \):

\[
\alpha = -3y^2 - z^2 \quad \text{and} \quad \beta = 2y \cdot (y^2 - z^2).
\]

- Now, we can apply shift by \( s \) and multiplication by \( a \).

- Thus, we arrive at the following algorithm for computing all possible “nice” cubic polynomials.
11. Resulting Algorithm for Computing All “Nice” Cubic Polynomials

• We use four arbitrary rational numbers \( t, y, s, \) and \( a; \) based on these numbers, we first compute
  \[ z = \frac{6yt}{3 - t^2}. \]

• Then, we compute the coefficients \( b, c, \) and \( d \) of the resulting “nice” polynomial (\( a \) we already know):
  \[ b = 3a \cdot s; \quad c = a \cdot (3s^2 - 3y^2 - z^2); \]
  \[ d = a \cdot (s^3 + 2y \cdot (y^2 - z^2)). \]

• These expressions cover almost all “nice” polynomials, with the exception of the following family:
  \[ b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3). \]
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13. Bibliography


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