Dealing with Uncertainties in Data Processing: from Probabilistic and Interval Uncertainty to Combination of Different Approaches, with Application to Geoinformatics, Bioinformatics, and Engineering

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1. General Problem of Data Processing under Uncertainty

- *Indirect measurements*: way to measure $y$ that are difficult (or even impossible) to measure directly.

- *Idea*: $y = f(x_1, \ldots, x_n)$

\[
\begin{array}{c}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n \\
\end{array}
\xrightarrow{f}
\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)
\]

- *Problem*: measurements are never 100% accurate: $\tilde{x}_i \neq x_i$ ($\Delta x_i \neq 0$) hence

\[
\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n) \neq y = f(x_1, \ldots, x_n).
\]

What are bounds on $\Delta y \overset{\text{def}}{=} \tilde{y} - y$?
2. Probabilistic and Interval Uncertainty

- **Traditional approach**: we know probability distribution for $\Delta x_i$ (usually Gaussian).
- **Where it comes from**: calibration using standard MI.
- **Problem**: calibration is not possible in:
  - fundamental science
  - manufacturing
- **Solution**: we know upper bounds $\Delta_i$ on $|\Delta x_i|$ hence
  $$x_i \in [\bar{x}_i - \Delta_i, \bar{x}_i + \Delta_i].$$
3. Interval Computations: A Problem

Given: an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) intervals \( x_i = [\underline{x}_i, \overline{x}_i] \).

Compute: the corresponding range of \( y \):
\[
[y, \overline{y}] = \{ f(x_1, \ldots, x_n) \mid x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n] \}.
\]

Fact: NP-hard even for quadratic \( f \).

Challenge: when are feasible algorithms possible?

Challenge: when computing \( y = [y, \overline{y}] \) is not feasible, find a good approximation \( Y \supseteq y \).
4. Alternative Approach: Maximum Entropy

- **Situation:** in many practical applications, it is very difficult to come up with the probabilities.

- **Traditional engineering approach:** use probabilistic techniques.

- **Problem:** many different probability distributions are consistent with the same observations.

- **Solution:** select one of these distributions – e.g., the one with the largest entropy.

- **Example – single variable:** if all we know is that \( x \in [x, \bar{x}] \), then MaxEnt leads to a uniform distribution.

- **Example – multiple variables:** different variables are independently distributed.
5. Limitations of Maximum Entropy Approach

• **Example**: simplest algorithm $y = x_1 + \ldots + x_n$.

• **Measurement errors**: $\Delta x_i \in [-\Delta, \Delta]$.

• **Analysis**: $\Delta y = \Delta x_1 + \ldots + \Delta x_n$.

• **Worst case situation**: $\Delta y = n \cdot \Delta$.

• **Maximum Entropy approach**: due to Central Limit Theorem, $\Delta y$ is $\approx$ normal, with $\sigma = \Delta \cdot \frac{\sqrt{n}}{\sqrt{3}}$.

• **Why this may be inadequate**: we get $\Delta \sim \sqrt{n}$, but due to correlation, it is possible that $\Delta = n \cdot \Delta \sim n \gg \sqrt{n}$.

• **Conclusion**: using a single distribution can be very misleading, especially if we want guaranteed results.

• **Examples**: high-risk application areas such as space exploration or nuclear engineering.
6. Linearization is Usually Possible

- In many practical situations, the errors $\Delta x_i$ are small, so we can ignore quadratic terms:

$$
\Delta y = \tilde{y} - y = f(\tilde{x}_1, \ldots, \tilde{x}_n) - f(x_1, \ldots, x_n) =
\begin{align*}
f(\tilde{x}_1, \ldots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n) \approx 
& c_1 \cdot \Delta x_1 + \ldots + c_n \cdot \Delta x_n, 
\end{align*}
$$

where $c_i \overset{\text{def}}{=} \frac{\partial f}{\partial x_i}(\tilde{x}_1, \ldots, \tilde{x}_n)$.

- For a linear function, the largest $\Delta y$ is obtained when each term $c_i \cdot \Delta x_i$ is the largest:

$$
\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n.
$$

- Due to the linearization assumption, we can estimate each partial derivative $c_i$ as

$$
c_i \approx \frac{f(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) - \tilde{y}}{h_i}.
$$
7. Linearization: Algorithm

To compute the range $y$ of $y$, we do the following.

- First, we apply the algorithm $f$ to the original estimates $\tilde{x}_1, \ldots, \tilde{x}_n$, resulting in the value $\tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)$.

- Second, for all $i$ from 1 to $n$,
  
  - we compute $f(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$ for some small $h_i$ and then
  
  - we compute
    
    $$c_i = \frac{f(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i + h_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) - \tilde{y}}{h_i}.$$ 

- Finally, we compute $\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n$ and the desired range $y = [\tilde{y} - \Delta, \tilde{y} + \Delta]$.

- **Problem**: we need $n + 1$ calls to $f$, and this is often too long.
8. Cauchy Deviate Method: Idea

- For large $n$, we can further reduce the number of calls to $f$ if we Cauchy distributions, w/pdf

$$\rho(z) = \frac{\Delta}{\pi \cdot (z^2 + \Delta^2)}.$$ 

- Known property of Cauchy transforms:
  - if $z_1, \ldots, z_n$ are independent Cauchy random variables w/parameters $\Delta_1, \ldots, \Delta_n$,
  - then $z = c_1 \cdot z_1 + \ldots + c_n \cdot z_n$ is also Cauchy distributed, w/parameter

$$\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n.$$ 

- This is exactly what we need to estimate interval uncertainty!
9. Cauchy Deviate Method: Towards Implementation

- To implement the Cauchy idea, we must answer the following questions:
  - how to simulate the Cauchy distribution; and
  - how to estimate the parameter $\Delta$ of this distribution from a finite sample.

- Simulation can be based on the functional transformation of uniformly distributed sample values:
  \[ \delta_i = \Delta_i \cdot \tan(\pi \cdot (r_i - 0.5)), \text{ where } r_i \sim U([0, 1]). \]

- To estimate $\Delta$, we can apply the Maximum Likelihood Method
  \( \rho(\delta^{(1)}) \cdot \rho(\delta^{(2)}) \cdot \ldots \cdot \rho(\delta^{(N)}) \rightarrow \max, \text{ i.e., solve} \)
  \[ \frac{1}{1 + \left( \frac{\delta^{(1)}}{\Delta} \right)^2} + \ldots + \frac{1}{1 + \left( \frac{\delta^{(N)}}{\Delta} \right)^2} = \frac{N}{2}. \]
10. Cauchy Deviates Method: Algorithm

- Apply $f$ to $\tilde{x}_i$; we get $\tilde{y} := f(\tilde{x}_1, \ldots, \tilde{x}_n)$.
- For $k = 1, 2, \ldots, N$, repeat the following:
  - use the standard RNG to draw $r_i^{(k)} \sim U([0, 1])$, $i = 1, 2, \ldots, n$;
  - compute Cauchy distributed values $c_i^{(k)} := \tan(\pi \cdot (r_i^{(k)} - 0.5))$;
  - compute $K := \max_i |c_i^{(k)}|$ and normalized errors $\delta_i^{(k)} := \Delta_i \cdot c_i^{(k)}/K$;
  - compute the simulated “actual values” $x_i^{(k)} := \tilde{x}_i - \delta_i^{(k)}$;
  - compute simulated errors of indirect measurement: $\delta^{(k)} := K \cdot \left( \tilde{y} - f \left( x_1^{(k)}, \ldots, x_n^{(k)} \right) \right)$;
- Compute $\Delta$ by applying the bisection method to solve the Maximum Likelihood equation.
11. Important Comment

- To avoid confusion, we should emphasize that:
  - in contrast to the Monte-Carlo solution for the probabilistic case,
  - the use of Cauchy distribution in the interval case is a computational trick,
  - it is not a truthful simulation of the actual measurement error $\Delta x_i$.

- Indeed:
  - we know that the actual value of $\Delta x_i$ is always inside the interval $[-\Delta_i, \Delta_i]$, but
  - a Cauchy distributed random attains values outside this interval as well.
12. Approximate Methods – Such As Linearization – Are Sometimes Not Sufficient

- In many application areas, it is sufficient to have an approximate estimate of \( y \).
- Sometimes, we need to guarantee that \( y \) does not exceed a certain threshold \( y_0 \). Examples:
  - in nuclear engineering, the temperatures and the neutron flows should not exceed the critical values;
  - a space ship lands on the planet and does not fly past it, etc.
- The only way to guarantee this is to have an interval \( \mathbf{Y} = [\underline{Y}, \overline{Y}] \) for which \( y \subseteq \mathbf{Y} \) and \( \overline{Y} \leq y_0 \).
- Such an interval is called an enclosure.
- Computing such an enclosure is one of the main tasks of interval computations.
13. Interval Computations: A Brief History

- **Origins:** Archimedes (Ancient Greece)

- **Modern pioneers:** Warmus (Poland), Sunaga (Japan), Moore (USA), 1956–59

- **First boom:** early 1960s.

- **First challenge:** taking interval uncertainty into account when planning spaceflights to the Moon.

- **Current applications** (sample):
  - design of elementary particle colliders: Berz, Kyoko (USA)
  - will a comet hit the Earth: Berz, Moore (USA)
  - robotics: Jaulin (France), Neumaier (Austria)
  - chemical engineering: Stadtherr (USA)

- **Problem:** compute the range
  
  \[ [\underline{y}, \overline{y}] = \{ f(x_1, \ldots, x_n) \mid x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n] \}. \]

- **Interval arithmetic:** for arithmetic operations \( f(x_1, x_2) \) (and for elementary functions), we have explicit formulas for the range.

- **Examples:** when \( x_1 \in \mathbf{x}_1 = [\underline{x}_1, \overline{x}_1] \) and \( x_2 \in \mathbf{x}_2 = [\underline{x}_2, \overline{x}_2] \), then:
  
  - The range \( \mathbf{x}_1 + \mathbf{x}_2 \) for \( x_1 + x_2 \) is \([\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]\).
  - The range \( \mathbf{x}_1 - \mathbf{x}_2 \) for \( x_1 - x_2 \) is \([\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2]\).
  - The range \( \mathbf{x}_1 \cdot \mathbf{x}_2 \) for \( x_1 \cdot x_2 \) is \([\underline{y}, \overline{y}]\), where
    
    \[
    \underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2);
    \]
    
    \[
    \overline{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2).
    \]
  - The range \( 1/\mathbf{x}_1 \) for \( 1/x_1 \) is \([1/\overline{x}_1, 1/\underline{x}_1]\) (if \( 0 \not\in \mathbf{x}_1 \)).
15. Straightforward Interval Computations: Example

- **Example:** \( f(x) = (x - 2) \cdot (x + 2), \ x \in [1, 2] \).

- **How will the computer compute it?**
  - \( r_1 := x - 2; \)
  - \( r_2 := x + 2; \)
  - \( r_3 := r_1 \cdot r_2. \)

- **Main idea:** perform the same operations, but with *intervals* instead of *numbers*:
  - \( r_1 := [1, 2] - [2, 2] = [-1, 0]; \)
  - \( r_2 := [1, 2] + [2, 2] = [3, 4]; \)
  - \( r_3 := [-1, 0] \cdot [3, 4] = [-4, 0]. \)

- **Actual range:** \( f(x) = [-3, 0]. \)

- **Comment:** this is just a toy example, there are more efficient ways of computing an enclosure \( Y \supseteq y \).
16. First Idea: Use of Monotonicity

- **Reminder:** for arithmetic, we had exact ranges.
- **Reason:** $+, -, \cdot$ are monotonic in each variable.
- **How monotonicity helps:** if $f(x_1, \ldots, x_n)$ is (non-strictly) increasing ($f \uparrow$) in each $x_i$, then
  \[ f(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n), f(\bar{x}_1, \ldots, \bar{x}_n)]. \]
- **Similarly:** if $f \uparrow$ for some $x_i$ and $f \downarrow$ for other $x_j$.
- **Fact:** $f \uparrow$ in $x_i$ if $\frac{\partial f}{\partial x_i} \geq 0$.
- **Checking monotonicity:** check that the range $[\underline{r}_i, \overline{r}_i]$ of $\frac{\partial f}{\partial x_i}$ on $x_i$ has $\underline{r}_i \geq 0$.
- **Differentiation:** by Automatic Differentiation (AD) tools.
- **Estimating ranges of $\frac{\partial f}{\partial x_i}$:** straightforward interval comp.
17. Monotonicity: Example

• **Idea:** if the range $[r_i, \bar{r}_i]$ of each $\frac{\partial f}{\partial x_i}$ on $x_i$ has $r_i \geq 0$, then

$$f(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n), f(\bar{x}_1, \ldots, \bar{x}_n)].$$

• **Example:** $f(x) = (x - 2) \cdot (x + 2)$, $x = [1, 2]$.

• **Case $n = 1$:** if the range $[r, \bar{r}]$ of $\frac{df}{dx}$ on $x$ has $r \geq 0$, then

$$f(x) = [f(x), f(\bar{x})].$$

• **AD:** $\frac{df}{dx} = 1 \cdot (x + 2) + (x - 2) \cdot 1 = 2x$.

• **Checking:** $[r, \bar{r}] = [2, 4]$, with $2 \geq 0$.

• **Result:** $f([1, 2]) = [f(1), f(2)] = [-3, 0]$.

• **Comparison:** this is the exact range.
18. Non-Monotonic Example

- Example: \( f(x) = x \cdot (1 - x), \ x \in [0, 1] \).
- How will the computer compute it?
  - \( r_1 := 1 - x; \)
  - \( r_2 := x \cdot r_1. \)

- Straightforward interval computations:
  - \( r_1 := [1, 1] - [0, 1] = [0, 1]; \)
  - \( r_2 := [0, 1] \cdot [0, 1] = [0, 1]. \)

- Actual range: \( \min, \max \) of \( f \) at \( x, \bar{x} \), or when \( \frac{df}{dx} = 0. \)
- Here, \( \frac{df}{dx} = 1 - 2x = 0 \) for \( x = 0.5 \), thus we:
  - compute \( f(0) = 0, \ f(0.5) = 0.25, \) and \( f(1) = 0, \) so
  - \( \underline{y} = \min(0, 0.25, 0) = 0, \ \bar{y} = \max(0, 0.25, 0) = 0.25. \)
- Resulting range: \( f(x) = [0, 0.25]. \)
19. Second Idea: Centered Form

- **Main idea:** Intermediate Value Theorem

\[ f(x_1, \ldots, x_n) = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i) \]

for some \( \chi_i \in x_i \).

- **Corollary:** \( f(x_1, \ldots, x_n) \in Y \), where

\[ Y = \tilde{y} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \cdot [-\Delta_i, \Delta_i]. \]

- **Differentiation:** by Automatic Differentiation (AD) tools.

- **Estimating the ranges of derivatives:**
  - if appropriate, by monotonicity, or
  - by straightforward interval computations, or
  - by centered form (more time but more accurate).
20. Centered Form: Example

- General formula:

\[ Y = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \cdot [-\Delta_i, \Delta_i]. \]

- Example: \( f(x) = x \cdot (1 - x), \ x = [0, 1]. \)

- Here, \( x = [\tilde{x} - \Delta, \tilde{x} + \Delta], \) with \( \tilde{x} = 0.5 \) and \( \Delta = 0.5. \)

- Case \( n = 1: \) \( Y = f(\tilde{x}) + \frac{df}{dx}(x) \cdot [-\Delta, \Delta]. \)

- AD: \( \frac{df}{dx} = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x. \)

- Estimation: we have \( \frac{df}{dx}(x) = 1 - 2 \cdot [0, 1] = [-1, 1]. \)

- Result: \( Y = 0.5 \cdot (1 - 0.5) + [-1, 1] \cdot [-0.5, 0.5] = 0.25 + [-0.5, 0.5] = [-0.25, 0.75]. \)

- Comparison: actual range \([0, 0.25],\) straightforward \([0, 1].\)
21. Third Idea: Bisection

- **Known:** accuracy $O(\Delta_i^2)$ of first order formula
  \[f(x_1, \ldots, x_n) = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i).\]

- **Idea:** if the intervals are too wide, we:
  - split one of them in half ($\Delta_i^2 \to \Delta_i^2/4$); and
  - take the union of the resulting ranges.

- **Example:** $f(x) = x \cdot (1 - x)$, where $x \in x = [0, 1]$.

- **Split:** take $x' = [0, 0.5]$ and $x'' = [0.5, 1]$.

- **1st range:** $1 - 2 \cdot x = 1 - 2 \cdot [0, 0.5] = [0, 1]$, so $f \uparrow$ and $f(x') = [f(0), f(0.5)] = [0, 0.25]$.

- **2nd range:** $1 - 2 \cdot x = 1 - 2 \cdot [0.5, 1] = [-1, 0]$, so $f \downarrow$ and $f(x'') = [f(1), f(0.5)] = [0, 0.25]$.

- **Result:** $f(x') \cup f(x'') = [0, 0.25] - \text{exact}$. 
22. Alternative Approach: Affine Arithmetic

- **So far:** we compute the range of $x \cdot (1 - x)$ by multiplying ranges of $x$ and $1 - x$.
- **We ignore:** that both factors depend on $x$ and are, thus, dependent.
- **Idea:** for each intermediate result $a$, keep an explicit dependence on $\Delta x_i = \tilde{x}_i - x_i$ (at least its linear terms).
- **Implementation:**
  \[
  a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + [a, \bar{a}].
  \]
- **We start:** with $x_i = \tilde{x}_i - \Delta x_i$, i.e.,
  \[
  \tilde{x}_i + 0 \cdot \Delta x_1 + \ldots + 0 \cdot \Delta x_{i-1} + (-1) \cdot \Delta x_i + 0 \cdot \Delta x_{i+1} + \ldots + 0 \cdot \Delta x_n + [0, 0].
  \]
- **Description:** $a_0 = \tilde{x}_i$, $a_i = -1$, $a_j = 0$ for $j \neq i$, and $[a, \bar{a}] = [0, 0]$. 
23. Affine Arithmetic: Operations

- **Representation:** \( a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + [a, \bar{a}]. \)

- **Input:** \( a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + a \) and \( b = b_0 + \sum_{i=1}^{n} b_i \cdot \Delta x_i + b. \)

- **Operations:** \( c = a \otimes b. \)

- **Addition:** \( c_0 = a_0 + b_0, \ c_i = a_i + b_i, \ c = a + b. \)

- **Subtraction:** \( c_0 = a_0 - b_0, \ c_i = a_i - b_i, \ c = a - b. \)

- **Multiplication:** \( c_0 = a_0 \cdot b_0, \ c_i = a_0 \cdot b_i + b_0 \cdot a_i, \)

\[
\begin{align*}
\sum_{i} a_i \cdot b_i \cdot [-\Delta_i, \Delta_i] & + \\
\sum_{i} a_i \cdot [\Delta_i] & + \\
\sum_{\substack{i \neq j}} a_i \cdot b_i \cdot [-\Delta_i, \Delta_i] & + \\
(\sum_{i} a_i \cdot [-\Delta_i, \Delta_i]) \cdot b & + \\
(\sum_{i} b_i \cdot [-\Delta_i, \Delta_i]) \cdot a & + a \cdot b.
\end{align*}
\]
24. Affine Arithmetic: Example

- **Example:** \( f(x) = x \cdot (1 - x), \ x \in [0, 1] \).
- Here, \( n = 1, \tilde{x} = 0.5, \) and \( \Delta = 0.5 \).
- How will the computer compute it?
  - \( r_1 := 1 - x; \)
  - \( r_2 := x \cdot r_1. \)
- **Affine arithmetic:** we start with \( x = 0.5 - \Delta x + [0, 0]; \)
  - \( r_1 := 1 - (0.5 - \Delta x) = 0.5 + \Delta x; \)
  - \( r_2 := (0.5 - \Delta x) \cdot (0.5 + \Delta x), \) i.e.,
    \[ r_2 = 0.25 + 0 \cdot \Delta x - [\Delta, \Delta]^2 = 0.25 + [-\Delta^2, 0]. \]
- **Resulting range:** \( y = 0.25 + [-0.25, 0] = [0, 0.25]. \)
- **Comparison:** this is the exact range.
25. Affine Arithmetic: Towards More Accurate Estimates

- *In our simple example:* we got the exact range.
- *In general:* range estimation is NP-hard.
- *Meaning:* a feasible (polynomial-time) algorithm will sometimes lead to excess width: \( Y \supset y \).
- *Conclusion:* affine arithmetic may lead to excess width.
- *Question:* how to get more accurate estimates?
- *First idea:* bisection.
- *Second idea* (Taylor arithmetic):
  - *affine arithmetic:* \( a = a_0 + \sum a_i \cdot \Delta x_i + a \);
  - *meaning:* we keep linear terms in \( \Delta x_i \);
  - *idea:* keep, e.g., quadratic terms
    \[
    a = a_0 + \sum a_i \cdot \Delta x_i + \sum a_{ij} \cdot \Delta x_i \cdot \Delta x_j + a.
    \]
26. Interval Computations vs. Affine Arithmetic: Comparative Analysis

- **Objective:** we want a method that computes a reasonable estimate for the range in reasonable time.

- **Conclusion – how to compare different methods:**
  - how accurate are the estimates, and
  - how fast we can compute them.

- **Accuracy:** affine arithmetic leads to more accurate ranges.

- **Computation time:**
  - *Interval arithmetic:* for each intermediate result \( a \), we compute two values: endpoints \( a \) and \( \bar{a} \) of \([a, \bar{a}]\).
  - *Affine arithmetic:* for each \( a \), we compute \( n + 3 \) values:
    \[
    a_0, a_1, \ldots, a_n, a, \bar{a}.
    \]

- **Conclusion:** affine arithmetic is \( \sim n \) times slower.
27. Solving Systems of Equations: Extending Known Algorithms to Situations with Interval Uncertainty

- **We have:** a system of equations $g_i(y_1, \ldots, y_n) = a_i$ with unknowns $y_i$;
- **We know:** $a_i$ with interval uncertainty: $a_i \in [\underline{a}_i, \overline{a}_i]$;
- **We want:** to find the corresponding ranges of $y_j$.
- **First case:** for exactly known $a_i$, we have an algorithm $y_j = f_j(a_1, \ldots, a_n)$ for solving the system.
- **Example:** system of linear equations.
- **Solution:** apply interval computations techniques to find the range $f_j([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n])$.
- **Better solution:** for specific equations, we often already know which ideas work best.
- **Example:** linear equations $A y = b$; $y$ is monotonic in $b$. 

28. Solving Systems of Equations When No Algorithm Is Known

- **Idea:**
  - parse each equation into elementary constraints, and
  - use interval computations to improve original ranges until we get a narrow range (= solution).

- **First example:** \( x - x^2 = 0.5, \ x \in [0, 1] \) (no solution).

- **Parsing:** \( r_1 = x^2, \ 0.5 (= r_2) = x - r_1 \).

- **Rules:** from \( r_1 = x^2 \), we extract two rules:
  
  (1) \( x \rightarrow r_1 = x^2 \);
  (2) \( r_1 \rightarrow x = \sqrt{r_1} \);

  from \( 0.5 = x - r_1 \), we extract two more rules:

  (3) \( x \rightarrow r_1 = x - 0.5 \);
  (4) \( r_1 \rightarrow x = r_1 + 0.5 \).
29. Solving Systems of Equations When No Algorithm Is Known: Example

- (1) \( r = x^2 \); (2) \( x = \sqrt{r} \); (3) \( r = x - 0.5 \); (4) \( x = r + 0.5 \).

- We start with: \( x = [0, 1] \), \( r = (\infty, \infty) \).

(1) \( r = [0, 1]^2 = [0, 1] \), so \( r_{\text{new}} = (\infty, \infty) \cap [0, 1] = [0, 1] \).

(2) \( x_{\text{new}} = \sqrt{[0, 1]} \cap [0, 1] = [0, 1] \) – no change.

(3) \( r_{\text{new}} = ([0, 1] - 0.5) \cap [0, 1] = [-0.5, 0.5] \cap [0, 1] = [0, 0.5] \).

(4) \( x_{\text{new}} = ([0, 0.5] + 0.5) \cap [0, 1] = [0.5, 1] \cap [0, 1] = [0.5, 1] \).

(1) \( r_{\text{new}} = [0.5, 1]^2 \cap [0, 0.5] = [0.25, 0.5] \).

(2) \( x_{\text{new}} = \sqrt{[0.25, 0.5]} \cap [0.5, 1] = [0.5, 0.71] \);
round \( a \) down \( \downarrow \) and \( \bar{a} \) up \( \uparrow \), to guarantee enclosure.

(3) \( r_{\text{new}} = ([0.5, 0.71] - 0.5) \cap [0.25, 5] = [0, 0.21] \cap [0.25, 0.5] \),
i.e., \( r_{\text{new}} = \emptyset \).

- Conclusion: the original equation has no solutions.
30. Solving Systems of Equations: 2nd Example

- **Example:** \( x - x^2 = 0, \ x \in [0, 1] \).
- **Parsing:** \( r_1 = x^2, \ 0 (= r_2) = x - r_1 \).
- **Rules:** (1) \( r = x^2 \); (2) \( x = \sqrt{r} \); (3) \( r = x \); (4) \( x = r \).
- **We start with:** \( x = [0, 1], \ r = (-\infty, \infty) \).
- **Problem:** after Rule 1, we’re stuck with \( x = r = [0, 1] \).
- **Solution:** bisect \( x = [0, 1] \) into \([0, 0.5]\) and \([0.5, 1]\).
- **For 1st subinterval:**
  - Rule 1 leads to \( r_{\text{new}} = [0, 0.5]^2 \cap [0, 0.5] = [0, 0.25] \);
  - Rule 4 leads to \( x_{\text{new}} = [0, 0.25] \);
  - Rule 1 leads to \( r_{\text{new}} = [0, 0.25]^2 = [0, 0.0625] \);
  - Rule 4 leads to \( x_{\text{new}} = [0, 0.0625] \); etc.
  - we converge to \( x = 0 \).
- **For 2nd subinterval:** we converge to \( x = 1 \).
31. Optimization: Extending Known Algorithms to Situations with Interval Uncertainty

- **Problem**: find $y_1, \ldots, y_m$ for which 
  $$g(y_1, \ldots, y_m, a_1, \ldots, a_m) \rightarrow \text{max}.$$ 

- **We know**: $a_i$ with interval uncertainty: $a_i \in [a_i, \bar{a}_i]$; 

- **We want**: to find the corresponding ranges of $y_j$. 

- **First case**: for exactly known $a_i$, we have an algorithm $y_j = f_j(a_1, \ldots, a_n)$ for solving the optimization problem. 

- **Example**: quadratic objective function $g$. 

- **Solution**: apply interval computations techniques to find the range $f_j([a_1, \bar{a}_1], \ldots, [a_n, \bar{a}_n])$. 

- **Better solution**: for specific $f$, we often already know which ideas work best.
32. Optimization When No Algorithm Is Known

- **Idea:** divide the original box \( x \) into subboxes \( b \).
- If \( \max_{x \in b} g(x) < g(x') \) for a known \( x' \), dismiss \( b \).
- **Example:** \( g(x) = x \cdot (1 - x) \), \( x = [0, 1] \).
- Divide into 10 (?) subboxes \( b = [0, 0.1], [0.1, 0.2], \ldots \)
- Find \( g(\tilde{b}) \) for each \( b \); the largest is \( 0.45 \cdot 0.55 = 0.2475 \).
- Compute \( G(b) = g(\tilde{b}) + (1 - 2 \cdot b) \cdot [-\Delta, \Delta] \).
- Dismiss subboxes for which \( \bar{Y} < 0.2475 \).
- **Example:** for \( [0.2, 0.3] \), we have
  \[
  0.25 \cdot (1 - 0.25) + (1 - 2 \cdot [0.2, 0.3]) \cdot [-0.05, 0.05].
  \]
- Here \( \bar{Y} = 0.2175 < 0.2475 \), so we dismiss \( [0.2, 0.3] \).
- **Result:** keep only boxes \( \subseteq [0.3, 0.7] \).
- **Further subdivision:** get us closer and closer to \( x = 0.5 \).
33. Case Study: Chip Design

- *Chip design:* one of the main objectives is to decrease the clock cycle.
- *Current approach:* uses worst-case (interval) techniques.
- *Problem:* the probability of the worst-case values is usually very small.
- *Result:* estimates are over-conservative – unnecessary over-design and under-performance of circuits.
- *Difficulty:* we only have *partial* information about the corresponding probability distributions.
- *Objective:* produce estimates valid for all distributions which are consistent with this information.
- *What we do:* provide such estimates for the clock time.
34. Estimating Clock Cycle: a Practical Problem

- **Objective:** estimate the clock cycle on the design stage.

- The clock cycle of a chip is constrained by the maximum path delay over all the circuit paths

\[ D \overset{\text{def}}{=} \max(D_1, \ldots, D_N). \]

- The path delay \( D_i \) along the \( i \)-th path is the sum of the delays corresponding to the gates and wires along this path.

- Each of these delays, in turn, depends on several factors such as:
  - the variation caused by the current design practices,
  - environmental design characteristics (e.g., variations in temperature and in supply voltage), etc.
35. Traditional (Interval) Approach to Estimating the Clock Cycle

- **Traditional approach:** assume that each factor takes the worst possible value.
- **Result:** time delay when all the factors are at their worst.
- **Problem:**
  - different factors are usually independent;
  - combination of worst cases is improbable.
- **Computational result:** current estimates are 30% above the observed clock time.
- **Practical result:** the clock time is set too high – chips are over-designed and under-performing.
36. Robust Statistical Methods Are Needed

- **Ideal case:** we know probability distributions.
- **Solution:** Monte-Carlo simulations.
- **In practice:** we only have *partial* information about the distributions of some of the parameters; usually:
  - the mean, and
  - some characteristic of the deviation from the mean
    - e.g., the interval that is guaranteed to contain possible values of this parameter.
- **Possible approach:** Monte-Carlo with several possible distributions.
- **Problem:** no guarantee that the result is a valid bound for all possible distributions.
- **Objective:** provide *robust* bounds, i.e., bounds that work for all possible distributions.
37. Towards a Mathematical Formulation of the Problem

- **General case:** each gate delay \( d \) depends on the difference \( x_1, \ldots, x_n \) between the actual and the nominal values of the parameters.

- **Main assumption:** these differences are usually small.

- Each path delay \( D_i \) is the sum of gate delays.

- **Conclusion:** \( D_i \) is a linear function: 
  \[
  D_i = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j
  \]
  for some \( a_i \) and \( a_{ij} \).

- The desired maximum delay \( D = \max_i D_i \) has the form
  \[
  D = F(x_1, \ldots, x_n) \overset{\text{def}}{=} \max_i \left( a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j \right). 
  \]
38. Towards a Mathematical Formulation of the Problem (cont-d)

- Known: maxima of linear function are exactly convex functions:

\[ F(\alpha \cdot x + (1 - \alpha) \cdot y) \leq \alpha \cdot F(x) + (1 - \alpha) \cdot F(y) \]

for all \( x, y \) and for all \( \alpha \in [0, 1] \);

- We know: factors \( x_i \) are independent;
  - we know distribution of some of the factors;
  - for others, we know ranges \([x_j, \bar{x}_j]\) and means \(E_j\).

- Given: a convex function \( F \geq 0 \) and a number \( \varepsilon > 0 \).

- Objective: find the smallest \( y_0 \) s.t. for all possible distributions, we have \( y \leq y_0 \) with the probability \( \geq 1 - \varepsilon \).
39. Additional Property: Dependency is Non-Degenerate

- **Fact**: sometimes, we learn additional information about one of the factors $x_j$.

- **Example**: we learn that $x_j$ actually belongs to a proper subinterval of the original interval $[x_j, \bar{x}_j]$.

- **Consequence**: the class $\mathcal{P}$ of possible distributions is replaced with $\mathcal{P}' \subset \mathcal{P}$.

- **Result**: the new value $y_0'$ can only decrease: $y_0' \leq y_0$.

- **Fact**: if $x_j$ is irrelevant for $y$, then $y_0' = y_0$.

- **Assumption**: irrelevant variables been weeded out.

- **Formalization**: if we narrow down one of the intervals $[x_j, \bar{x}_j]$, the resulting value $y_0$ decreases: $y_0' < y_0$. 
40. Formulation of the Problem

**GIVEN:**
- \( n, k \leq n, \varepsilon > 0; \)
- a convex function \( y = F(x_1, \ldots, x_n) \geq 0; \)
- \( n - k \) cdfs \( F_j(x), k + 1 \leq j \leq n; \)
- intervals \( x_1, \ldots, x_k, \) values \( E_1, \ldots, E_k, \)

**TAKE:** all joint probability distributions on \( R^n \) for which:
- all \( x_i \) are independent,
- \( x_j \in x_j, E[x_j] = E_j \) for \( j \leq k, \) and
- \( x_j \) have distribution \( F_j(x) \) for \( j > k. \)

**FIND:** the smallest \( y_0 \) s.t. for all such distributions,
\( F(x_1, \ldots, x_n) \leq y_0 \) with probability \( \geq 1 - \varepsilon. \)

**WHEN:** the problem is *non-degenerate* – if we narrow down one of the intervals \( x_j, \) \( y_0 \) decreases.
41. Main Result and How We Can Use It

- **Result:** $y_0$ is attained when for each $j$ from 1 to $k$,
  - $x_j = \bar{x}_j$ with probability $p_j = \frac{\bar{x}_j - E_j}{\bar{x}_j - x_j}$, and
  - $x_j = \bar{x}_j$ with probability $\bar{p}_j = \frac{E_j - x_j}{\bar{x}_j - x_j}$.

- **Algorithm:**
  - simulate these distributions for $x_j$, $j < k$;
  - simulate known distributions for $j > k$;
  - use the simulated values $x^{(s)}_j$ to find
    \[ y^{(s)} = F(x^{(s)}_1, \ldots, x^{(s)}_n); \]
  - sort $N$ values $y^{(s)}$: $y(1) \leq y(2) \leq \ldots \leq y(N_i)$;
  - take $y(N_i \cdot (1 - \varepsilon))$ as $y_0$. 
42. Comment about Monte-Carlo Techniques

- *Traditional belief:* Monte-Carlo methods are inferior to analytical:
  - they are approximate;
  - they require large computation time;
  - simulations for *several* distributions, may mis-calculate the (desired) maximum over *all* distributions.

- *We proved:* the value corresponding to the selected distributions indeed provide the desired maximum value $y_0$.

- *General comment:*
  - justified Monte-Carlo methods often lead to *faster* computations than analytical techniques;
  - example: multi-D integration – where Monte-Carlo methods were originally invented.
43. Comment about Non-Linear Terms

- **Reminder**: in the above formula \( D_i = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j \), we ignored quadratic and higher order terms in the dependence of each path time \( D_i \) on parameters \( x_j \).

- **In reality**: we may need to take into account some quadratic terms.

- **Idea behind possible solution**: it is known that the max \( D = \max_i D_i \) of convex functions \( D_i \) is convex.

- **Condition when this idea works**: when each dependence \( D_i(x_1, \ldots, x_k, \ldots) \) is still convex.

- **Solution**: in this case,
  - the function function \( D \) is still convex,
  - hence, our algorithm will work.
44. Conclusions

- **Problem of chip design**: decrease the clock cycle.
- **How this problem is solved now**: by using worst-case (interval) techniques.
- **Limitations of this solution**: the probability of the worst-case values is usually very small.
- **Consequence**: estimates are over-conservative, hence over-design and under-performance of circuits.
- **Objective**: find the clock time as $y_0$ s.t. for the actual delay $y$, we have $\text{Prob}(y > y_0) \leq \varepsilon$ for given $\varepsilon > 0$.
- **Difficulty**: we only have partial information about the corresponding distributions.
- **What we have described**: a general technique that allows us, in particular, to compute $y_0$. 
45. Combining Interval and Probabilistic Uncertainty: General Case

• *Problem*: there are many ways to represent a probability distribution.

• *Idea*: look for an objective.

• *Objective*: make decisions $E_x[u(x, a)] \rightarrow \max_a$.

• *Case 1*: smooth $u(x)$.

• *Analysis*: we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \ldots$

• *Conclusion*: we must know moments to estimate $E[u]$.

• *Case of uncertainty*: interval bounds on moments.

• *Case 2*: threshold-type $u(x)$.

• *Conclusion*: we need cdf $F(x) = \text{Prob}(\xi \leq x)$.

• *Case of uncertainty*: p-box $[\underline{F}(x), \overline{F}(x)]$. 
46. Extension of Interval Arithmetic to Probabilistic Case: Successes

- **General solution**: parse to elementary operations $+$, $-$, $\cdot$, $1/x$, max, min.

- Explicit formulas for arithmetic operations known for intervals, for p-boxes $\mathbf{F}(x) = [\underline{F}(x), \bar{F}(x)]$, for intervals $+$ 1st moments $E_i \overset{\text{def}}{=} E[x_i]$:

\[
\begin{align*}
  x_1, E_1 & \quad \xrightarrow{f} \quad y, E \\
  x_2, E_2 & \quad \xrightarrow{f} \quad y, E \\
  \ldots & \quad \xrightarrow{f} \quad y, E \\
  x_n, E_n & \quad \xrightarrow{f} \quad y, E
\end{align*}
\]
47. Successes (cont-d)

- Easy cases: +, −, product of independent $x_i$.

- Example of a non-trivial case: multiplication $y = x_1 \cdot x_2$, when we have no information about the correlation:

  - $\bar{E} = \max(p_1+p_2-1, 0) \cdot \bar{x}_1 \cdot \bar{x}_2 + \min(p_1, 1-p_2) \cdot \bar{x}_1 \cdot \bar{x}_2 + \min(1-p_1, p_2) \cdot x_1 \cdot x_2 + \max(1-p_1-p_2, 0) \cdot x_1 \cdot x_2$;
  
  - $\overline{E} = \min(p_1, p_2) \cdot \bar{x}_1 \cdot \bar{x}_2 + \max(p_1-p_2, 0) \cdot \bar{x}_1 \cdot \bar{x}_2 + \max(p_2-p_1, 0) \cdot x_1 \cdot x_2 + \min(1-p_1, 1-p_2) \cdot x_1 \cdot x_2$,

where $p_i \overset{\text{def}}{=} (E_i - x_i)/(\bar{x}_i - x_i)$. 
48. Challenges

- intervals + 2nd moments:

\[
\begin{align*}
\mathbf{x}_1, E_1, V_1 \\
\mathbf{x}_2, E_2, V_2 \\
\vdots \\
\mathbf{x}_n, E_n, V_n
\end{align*}
\]

\[
\xrightarrow{f} y, E, V
\]

- moments + p-boxes; e.g.:

\[
\begin{align*}
E_1, F_1(x) \\
E_2, F_2(x) \\
\vdots \\
E_n, F_n(x)
\end{align*}
\]

\[
\xrightarrow{f} E, F(x)
\]
49. Case Study: Bioinformatics

- **Practical problem:** find genetic difference between cancer cells and healthy cells.
- **Ideal case:** we directly measure concentration $c$ of the gene in cancer cells and $h$ in healthy cells.
- **In reality:** difficult to separate.
- **Solution:** we measure $y_i \approx x_i \cdot c + (1 - x_i) \cdot h$, where $x_i$ is the percentage of cancer cells in $i$-th sample.
- **Equivalent form:** $a \cdot x_i + h \approx y_i$, where $a \overset{\text{def}}{=} c - h$. 
50. Case Study: Bioinformatics (cont-d)

- **If we know \( x_i \) exactly:** Least Squares Method

\[
\sum_{i=1}^{n} (a \cdot x_i + h - y_i)^2 \to \min_{a,h}, \text{ hence } a = \frac{C(x,y)}{V(x)} \text{ and } h = E(y) - a \cdot E(x), \text{ where } E(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i,
\]

\[
V(x) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x))^2,
\]

\[
C(x,y) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x)) \cdot (y_i - E(y)).
\]

- **Interval uncertainty:** experts manually count \( x_i \), and only provide interval bounds \( x_i \), e.g., \( x_i \in [0.7, 0.8] \).

- **Problem:** find the range of \( a \) and \( h \) corresponding to all possible values \( x_i \in [\underline{x}_i, \overline{x}_i] \).
51. General Problem

- **General problem:**
  - we know intervals \( x_1 = [x_{1}, \bar{x}_1], \ldots, x_n = [x_{n}, \bar{x}_n], \)
  - compute the range of \( E(x) = \frac{1}{n} \sum_{i=1}^{n} x_i, \) population variance \( V = \frac{1}{n} \sum_{i=1}^{n} (x_i - E(x))^2, \) etc.

- **Difficulty:** NP-hard even for variance.

- **Known:**
  - efficient algorithms for \( V, \)
  - efficient algorithms for \( V \) and \( C(x, y) \) for reasonable situations.

- **Bioinformatics case:** find intervals for \( C(x, y) \) and for \( V(x) \) and divide.
52. Case Study: Detecting Outliers

- In many application areas, it is important to detect outliers, i.e., unusual, abnormal values.
- In medicine, unusual values may indicate disease.
- In geophysics, abnormal values may indicate a mineral deposit (or an erroneous measurement result).
- In structural integrity testing, abnormal values may indicate faults in a structure.
- **Traditional engineering approach**: a new measurement result \( x \) is classified as an outlier if \( x \notin [L, U] \), where
  \[
  L \overset{\text{def}}{=} E - k_0 \cdot \sigma, \quad U \overset{\text{def}}{=} E + k_0 \cdot \sigma,
  \]
  and \( k_0 > 1 \) is pre-selected.
- **Comment**: most frequently, \( k_0 = 2, 3, \) or 6.
53. Outlier Detection Under Interval Uncertainty: A Problem

- In some practical situations, we only have intervals \( x_i = [x_i, \bar{x}_i] \).
- Different \( x_i \in x_i \) lead to different intervals \([L, U]\).
- A possible outlier: outside some \( k_0 \)-sigma interval.
- Example: structural integrity – not to miss a fault.
- A guaranteed outlier: outside all \( k_0 \)-sigma intervals.
- Example: before a surgery, we want to make sure that there is a micro-calcification.
- A value \( x \) is a possible outlier if \( x \notin [\bar{L}, \bar{U}] \).
- A value \( x \) is a guaranteed outlier if \( x \notin [L, U] \).
- Conclusion: to detect outliers, we must know the ranges of \( L = E - k_0 \cdot \sigma \) and \( U = E + k_0 \cdot \sigma \).
54. **Outlier Detection Under Interval Uncertainty: A Solution**

- **We need:** to detect outliers, we must compute the ranges of \( L = E - k_0 \cdot \sigma \) and \( U = E + k_0 \cdot \sigma \).
- **We know:** how to compute the ranges \( E \) and \([\sigma, \bar{\sigma}]\) for \( E \) and \( \sigma \).
- **Possibility:** use interval computations to conclude that \( L \in E - k_0 \cdot [\sigma, \bar{\sigma}] \) and \( L \in E + k_0 \cdot [\sigma, \bar{\sigma}] \).
- **Problem:** the resulting intervals for \( L \) and \( U \) are wider than the actual ranges.
- **Reason:** \( E \) and \( \sigma \) use the same inputs \( x_1, \ldots, x_n \) and are hence not independent from each other.
- **Practical consequence:** we miss some outliers.
- **Desirable:** compute *exact* ranges for \( L \) and \( U \).
- **Application:** detecting outliers in gravity measurements.
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56. **Fuzzy Computations: A Problem**

\[ \begin{array}{c}
\mu_1(x_1) \\
\mu_2(x_2) \\
\vdots \\
\mu_n(x_n)
\end{array} \xrightarrow{f} \mu = f(\mu_1, \ldots, \mu_n) \]

- **Given:** an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) fuzzy numbers \( \mu_i(x_i) \).

- **Compute:** \( \mu(y) = \max_{x_1, \ldots, x_n: f(x_1, \ldots, x_n) = y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)) \).

- **Motivation:** \( y \) is a possible value of \( Y \leftrightarrow \exists x_1, \ldots, x_n \) s.t. each \( x_i \) is a possible value of \( X_i \) and \( f(x_1, \ldots, x_n) = y \).

- **Details:** “and” is \( \min \), \( \exists \) (“or”) is \( \max \), hence

\[
\mu(y) = \max_{x_1, \ldots, x_n} \min(\mu_1(x_1), \ldots, \mu_n(x_n), t(f(x_1, \ldots, x_n) = y)),
\]

where \( t(\text{true}) = 1 \) and \( t(\text{false}) = 0 \).
57. Fuzzy Computations: Reduction to Interval Computations

• **Problem (reminder):**
  - **Given:** an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) fuzzy numbers \( X_i \) described by membership functions \( \mu_i(x_i) \).
  - **Compute:** \( Y = f(X_1, \ldots, X_n) \), where \( Y \) is defined by Zadeh’s extension principle:
    \[
    \mu(y) = \max_{x_1, \ldots, x_n : f(x_1, \ldots, x_n) = y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)).
    \]

• **Idea:** represent each \( X_i \) by its \( \alpha \)-cuts
  \[
  X_i(\alpha) = \{x_i : \mu_i(x_i) \geq \alpha\}.
  \]

• **Advantage:** for continuous \( f \), for every \( \alpha \), we have
  \[
  Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)).
  \]

• **Resulting algorithm:** for \( \alpha = 0, 0.1, 0.2, \ldots, 1 \) apply interval computations techniques to compute \( Y(\alpha) \).
58. Proof of the Result about Chips

- Let us fix the optimal distributions for $x_2, \ldots, x_n$; then,
  \[ \text{Prob}(D \leq y_0) = \sum_{(x_1, \ldots, x_n): D(x_1, \ldots, x_n) \leq y_0} p_1(x_1) \cdot p_2(x_2) \cdot \ldots \]

- So, \( \text{Prob}(D \leq y_0) = \sum_{i=0}^{N} c_i \cdot q_i \), where \( q_i \overset{\text{def}}{=} p_1(v_i) \).

- Restrictions: \( q_i \geq 0, \sum_{i=0}^{N} q_i = 1, \) and \( \sum_{i=0}^{N} q_i \cdot v_i = E_1 \).

- Thus, the worst-case distribution for $x_1$ is a solution to the following linear programming (LP) problem:

  Minimize \( \sum_{i=0}^{N} c_i \cdot q_i \) under the constraints \( \sum_{i=0}^{N} q_i = 1 \) and

  \( \sum_{i=0}^{N} q_i \cdot v_i = E_1, q_i \geq 0, \quad i = 0, 1, 2, \ldots, N. \)
59. Proof of the Result about Chips (cont-d)

- **Minimize**: $\sum_{i=0}^{N} c_i \cdot q_i$ under the constraints $\sum_{i=0}^{N} q_i = 1$ and $\sum_{i=0}^{N} q_i \cdot v_i = E_1$, $q_i \geq 0$, $i = 0, 1, 2, \ldots, N$.

- **Known**: in LP with $N + 1$ unknowns $q_0, q_1, \ldots, q_N$, $\geq N + 1$ constraints are equalities.

- **In our case**: we have 2 equalities, so at least $N - 1$ constraints $q_i \geq 0$ are equalities.

- Hence, no more than 2 values $q_i = p_1(v_i)$ are non-0.

- If corresponding $v$ or $v'$ are in $(\underline{x}_1, \overline{x}_1)$, then for $[v, v'] \subset \underline{x}_1$ we get the same $y_0$ – in contradiction to non-degeneracy.

- Thus, the worst-case distribution is located at $\underline{x}_1$ and $\overline{x}_1$.

- The condition that the mean of $x_1$ is $E_1$ leads to the desired formulas for $\underline{p}_1$ and $\overline{p}_1$. 