How to Bargain: An Interval Approach

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1. Need to Bargain: Typical Situations

- If you want to *buy* a house with a list price $\bar{a}$, what offer $a$ should you make?

- If you are already in a negotiating process,
  - your previous offer was $a$,
  - the seller’s last offer was $\bar{a} > a$,
  - what next offer $a$ should you make?

- If you want to *sell* a house, and a potential buyer made an offer $a$, what counter-offer $a$ should you make?

- If you are already in a negotiating process,
  - your previous offer was $\bar{a}$,
  - the buyer’s related counter-offer was $a < \bar{a}$,
  - what next offer $a$ should you make?
2. Need to Bargain: Other Examples

Other examples of bargaining:

- negotiating a *bankruptcy* deal:
  - if the original debt was $\bar{a}$, and
  - the company going bankrupt is offering to pay an amount $a < \bar{a}$,
  - what is a reasonable next offer?

- negotiating for a *salary* with a new hire;

- negotiating between an employer and an *insurance* company for the best way to provide insurance to the employees;

- in an *auction*, when the previous bid was $a$, what next bid should you make?
3. Commonsense Solutions

- A usual advise for the first offer is to offer a portion \( a = k \cdot \bar{a} \) of the asking price, for some \( k \in (0, 1) \).

- The exact value of the coefficient \( k \) depends on the situation:
  - when buying a house in the US, 70-80\% is usually appropriate;
  - when buying a souvenir in an oriental bazaar, 1/3 (or even 1/4) may be appropriate.

- A usual advise to use in the middle of negotiations is, e.g., to split the difference, i.e., to select \( a = (\underline{a} + \bar{a})/2 \).

- In most cases, the recommended offer \( a \) is a linear function of the bounds \( \underline{a} \) and \( \bar{a} \).
4. Usual Game Theory Approach to Bargaining: Reminder

- In general, economic situations like this, with conflict of interest, are handled by game theory.

- Traditional game-theoretic solution concepts:
  - select the set of reasonable outcome, and
  - leave the exact choice to the participants’ bargaining skills.

- Nash’s bargaining solution:
  - the first game-theoretic bargaining solution was proposed by the Nobelist John Nash;
  - he proposed to select an alternative with the largest product of utility gains;
  - under reasonable assumptions, this leads to a linear function of the bounds $a$ and $\bar{a}$. 
5. From Nash’s Bargaining Solution to Actual Bargaining: Successes and Limitations

- **Problem**: Nash’s bargaining solution does not explain how exactly we should bargain.

- **Solution**: a more sophisticated game-theoretic analysis leads to bargaining recommendations \( a = f(a, \bar{a}) \).

- **Details**: the resulting function \( f \) is linear.

- **Reminder**: there are many solution concepts and utility functions.

- **Problem**: under some other concepts and/or assumptions, we get a non-linear recommendation \( a = f(a, \bar{a}) \).

- **Objective**: recommendations which do not depend:
  
  - on the specific (and somewhat arbitrary) choice of a solution concept and/or
  
  - on difficult-to-verify assumptions about utility f-s.
6. First Offer Problem: Analysis

• **Reminder:** produce an offer \( a' = f(a) \) based on the asking price \( a \geq 0 \).

• **Situation:** we want to buy (or sell) two houses at the same time, with asking prices \( a \) and \( b \).

• If we treat these houses as *separate purchases*, we should offer \( f(a) \) for the 1st house and \( f(b) \) for the 2nd house.

• Thus, the total amount of the offer is \( f(a) + f(b) \).

• If we view the two houses as a *single purchase*, with the initial price \( t = a + b \), we offer \( f(t) = f(a + b) \).

• It makes sense to require that the total amount offered should not depend on how we treat the situation:

\[
f(a + b) = f(a) + f(b).
\]
7. First Offer Problem: Result

**Definition.** By a unary recommendation function, we mean a function \( f : \mathbb{R} \to \mathbb{R} \) that maps every non-negative number \( a \) into a non-negative value \( f(a) \) for which

\[
f(a + b) = f(a) + f(b).
\]

**Proposition.** Every unary recommendation function has the form \( f(a) = k \cdot a \) for some real number \( k \geq 0 \).

**Comment.** The choice of \( k \) depends on whether we consider a buyer or a seller problem:

- for a buyer, \( f(\bar{a}) \leq \bar{a} \), so \( k \leq 1 \);
- for a seller, \( f(\bar{a}) \geq \bar{a} \), so \( k \geq 1 \).
8. Selecting an Offer in the Middle of a Bargaining Process: Analysis

- **Reminder:** produce an offer \( a = f(a, \overline{a}) \) based on the current offers \( a < \overline{a} \).

- **Situation:** we want to buy (or sell) two houses at the same time, with offers \( a < \overline{a} \) and \( b < \overline{b} \).

- If we treat these houses as *separate purchases*, we should offer \( f(a, \overline{a}) \) for the 1st house and \( f(b, \overline{b}) \) for the 2nd.

- Thus, the total amount of the offer is \( f(a, \overline{a}) + f(b, \overline{b}) \).

- If we view the houses as a *single purchase*, with the current offers \( t \overset{\text{def}}{=} a + b < \overline{t} \overset{\text{def}}{=} \overline{a} + \overline{b} \), we offer \( f(t, \overline{t}) \).

- It makes sense to require that the total amount offered should not depend on how we treat the situation:

\[
f(a + b, \overline{a} + \overline{b}) = f(a, \overline{a}) + f(b, \overline{b}).
\]
9. Selecting an Offer in the Middle of a Bargaining Process: Result

Definition. By a binary recommendation function, we mean a function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) that maps every pair \((a, \bar{a})\) of non-negative numbers with \( a \leq \bar{a} \) into a non-negative value \( f(a, \bar{a}) \in [a, \bar{a}] \) and for which

\[
f(a + b, \bar{a} + \bar{b}) = f(a, \bar{a}) + f(b, \bar{b}).
\]

Proposition. Every binary recommendation function has the form

\[
f(a, \bar{a}) = k \cdot \bar{a} + (1 - k) \cdot a
\]

for some real number \( k \in [0, 1] \).
10. Relation to Intervals and Interval Computations

- The pair \((a, \overline{a})\) of non-negative numbers with \(a \leq \overline{a}\) represents an interval \(a = [a, \overline{a}]\).
- For intervals, addition is defined as
  \[a + b \overset{\text{def}}{=} \{a + b : a \in a, b \in b\}\].
- Known result: this leads to
  \[[a, \overline{a}] + [b, \overline{b}] = [a + b, \overline{a} + \overline{b}]\].
- Thus, in interval terms, the additivity requirement
  \[f(a + b, \overline{a} + \overline{b}) = f(a, \overline{a}) + f(b, \overline{b})\]
  takes the form
  \[f(a + b) = f(a) + f(b)\].
11. Relation to Decision Making Under Uncertainty

- Suppose that we only know that the price of an object $A$ is between $a$ and $\bar{a}$.
- What is a reasonable price $a = f(a, \bar{a})$ to pay for $A$?
- If we buy two objects, the fair price should not depend on whether we consider them separately or together:

$$f(a + b) = f(a) + f(b).$$

- Thus, according to our proposition, the fair price is $f(a, \bar{a}) = k \cdot \bar{a} + (1 - k) \cdot a$.
- This is the optimism-pessimism criterion proposed by another Nobelist L. Hurwicz:
  - $k = 1$ and $a = \bar{a}$ is the most optimistic case;
  - $k = 0$ and $a = a$ is the most pessimistic case.
12. Should Recommendation Depend on Pre-History?

**Definition.** Let $T$ be a positive integer. By a $T$-ary recommendation function, we mean a function $f$ that maps every tuple $(a_0, \ldots, a_T)$ of $T+1$ non-negative intervals satisfying the condition $a_{t+1} \subseteq a_t$ into a non-negative value

$$f(a_0, \ldots, a_T) \in a_T$$

for which

$$f(a_0 + b_0, \ldots, a_T + b_T) = f(a_0, \ldots, a_T) + f(b_0, \ldots, b_T).$$

**Proposition.** For every $T$, every $T$-ary recommendation function depends only on $a_T$: $f(a_0, \ldots, a_T) = f(a_T)$.

**Comment.** The resulting recommendation depends, Markov-process-style, only on the latest offer $a_T$. 
13. Proof: Main Idea

- The above result seems counter-intuitive, so we show the proof to make it more convincing.
- Due to additivity, the function $f(a_0, \ldots, a_T)$ linearly depends on all its variables.
- Thus,

$$f(a_0, \ldots, a_T) = \sum_{t=0}^{T} \ell_t \cdot a_t + \sum_{t=0}^{T} u_t \cdot \bar{a}_t$$

for some coefficients $\ell_t$ and $u_t$.
- We require that $f(a_0, \ldots, a_T) \in a_T$ for all $a_0, \ldots, a_T$.
- For the case when $a_T$ is a degenerate interval $a_T = [a_T, a_T]$, this means that $f(a_0, \ldots, a_T) = a_T$.
- Thus, the coefficients $\ell_t$ and $u_t$ corresponding to $t < T$ should be equal to 0. Q.E.D.
14. Previous Derivation of Hurwicz F-la: Reminder

- We derived the Hurwicz formula from the requirement the \( f(a, \bar{a}) \) be shift- and scale-invariant.
- Shift-invariance means that:
  - if we add a thing of a fixed cost \( c \) to the object,
  - the recommended price should increase by this price \( c \):
    \[
    f(a + c, \bar{a} + c) = f(a, \bar{a}) + c.
    \]
- Shift-invariance is a particular case of additivity.
- Scale-invariance means that:
  - if we use different (\( \lambda \) times smaller) monetary units,
  - recommendations should stay the same:
    \[
    f(\lambda \cdot a, \lambda \cdot \bar{a}) = \lambda \cdot f(a, \bar{a}).
    \]
- Similarly, shift- and scale-invariance imply the bargain formula \( f(a, \bar{a}) = k \cdot \bar{a} + (1 - k) \cdot a \).
15. Previous Derivation Cannot Be Extended to the Analysis of Dependence on Pre-History

- We derived the formulas for the bargaining from additivity;
- For $T = 0$ (no dependence on pre-history), the same formula can be derived from shift- and scale-invariance.
- However, already for $T = 1$, shift- and scale-invariance is not sufficient.
- **Counter-example:**
  - the recommendation
    \[
    f([a_0, \overline{a}_0], [a_1, \overline{a}_1]) = \max \left( a_1, \min \left( \overline{a}_1, \frac{a_0 + \overline{a}_0 + a_1 + \overline{a}_1}{4} \right) \right)
    \]
    is shift- and scale-invariant;
  - however, this recommendation depends on $a_0$ as well: e.g.,
    \[
    f([0, 3], [1, 2]) = 1.5 \neq f([0, 4], [1, 2]) = 1.75.
    \]
16. Application to Auctions

We can consider the dependence of the next bid on the previous bids $a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_T$. The result is different.

**Definition.** Let $T$ be a positive integer. By a $T$-ary auction recommendation function, we mean a function $f$ that maps every tuple $(a_0, \ldots, a_T)$ of $T + 1$ non-negative numbers satisfying the condition $a_t \leq a_{t+1}$ into a non-negative value $f(a_0, \ldots, a_T) \geq a_T$ for which

$$f(a_0 + b_0, \ldots, a_T + b_T) = f(a_0, \ldots, a_T) + f(b_0, \ldots, b_T).$$

**Proposition.** For every $T$, every $T$-ary recommendation function has the form

$$f(a_0, \ldots, a_T) = c_0 \cdot a_0 + \sum_{t=1}^{T} c_t \cdot (a_t - a_{t-1})$$

for some values $c_0 \geq 1, c_1 \geq 1, \ldots, c_T \geq 1$. 
17. How to Find the Equilibrium

• As a result of the negotiations process, we get converging offers and counter-offers.

• In the limit, we reach an “equilibrium” – the price to which both the buyer and the seller agree.

• We consider the cases that start with the seller’s offer, so \([a(0), \bar{a}(0)] = [0, \bar{a}_0]\).

• On each iteration, the buyer offers
  \[a_t = k_b \cdot \bar{a}_{t-1} + (1 - k_b) \cdot a_{t-1}.\]

• Now, the seller has an interval \([a_t, \bar{a}_{t-1}]\), and selects
  \[\bar{a}_t = k_s \cdot \bar{a}_{t-1} + (1 - k_s) \cdot a_t.\]

• One can prove that the process converges, and the resulting equilibrium is
  \[a = \frac{1 - k}{1 - k + k_b} \cdot \bar{a}_0,\] where
  \[k \overset{\text{def}}{=} k_s + k_b - k_s \cdot k_b.\]
18. What is the Purpose of Negotiations in the First Place?

- **Question:** the negotiation ends up with a single value $a$, so why not come up with this value right away?

- **Answer:** some buyers may value their time more and stop negotiations earlier.

- **Analysis** shows that at $t$-th iteration, the seller’s price is $\overline{a}_t = a + \Delta \cdot k_0^t$, where $\Delta \overset{\text{def}}{=} \overline{a}_0 - a$ and $k_0 \overset{\text{def}}{=} k - k_b$.

- **The buyer’s loss** at moment $t$ is $\Delta \cdot k_0^t + w_b \cdot t$, where $w_b$ is the cost of his/her time.

- **Recommendation:** the buyer stops when the loss is minimal:
  \[
  t = \frac{\ln(w_b) - \ln(\Delta) - \ln(|\ln(k_0)|)}{\ln(k_0)}.
  \]

- **Conclusion:** the more the buyer values his/her time, the earlier the buyer stops bargaining.
19. Acknowledgments

- This work was supported in part by NSF grants:
  - Cyber-ShARE Center of Excellence (HRD-0734825),
  - Computing Alliance of Hispanic-Serving Institutions CAHSI (CNS-0540592),
and by NIH Grant 1 T36 GM078000-01.