Prediction in Econometrics: Towards Mathematical Justification of Simple (and Successful) Heuristics

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1. Prediction is Important

- Prediction (forecasting) is of upmost importance in economics and finance.
- If we can accurately predict the future prices, then we can get the largest return on investment.
- Vice versa:
  - if we make decisions based on the wrong predictions,
  - then our financial investments collapse,
  - and the manufacturing plants that we built are non-profitable and thus idle.
- Many successful (semi-)heuristic methods have been proposed to predict economic and financial processes.
2. Need to Justify Heuristic Strategies

- The success of prediction heuristics leads to a conjecture that these heuristics have a theoretical justification.

- In general, when we have a theoretical justification, it helps:
  - we can use the corresponding theory to fine-tune the method, and
  - we can get a clearer understanding of when the method is efficient and when it is not efficient.

- In this paper, we justify two heuristics:
  - of an intuitive exponential smoothing procedure, that predicts slowly changing processes, and
  - of a seemingly counter-intuitive idea of an increase in volatility as a predictor of trend reversal.
3. First Result: Prediction of Slowly Changing Processes

- **Problem**: based on the past observations $x_1, \ldots, x_T$ ($x_1$ most recent), predict the future value $x_0$.

- **In other words**: we need a predictor function $x_0 \approx F(x_1, \ldots, x_T)$.

- **Continuity**: if $x_i \approx x'_i$, then $F(x_1, \ldots, x_T) \approx F(x'_1, \ldots, x'_T)$.

- **Motivation**: we predict based on measurement results, and they are never absolutely accurate.

- **Additivity**: $F(x_1^{(1)} + x_1^{(2)}, \ldots) = F(x_1^{(1)}, \ldots) + F(x_1^{(2)}, \ldots)$.

- **Motivation**: we can predict stocks $x_0^{(1)}$ and bonds $x_0^{(2)}$, or we can predict value of the whole portfolio $x_0^{(1)} + x_0^{(2)}$.

- **Conclusion**: we must consider linear predictors $F(x_1, \ldots, x_T) = \sum_{t=1}^{T} f_t \cdot x_t$. 
4. From Finite to Infinite Time

- The actual number of observed values is always finite.
- However, in many cases, we have very long time series (e.g., daily for many years).
- In real life, the influence of remote events is small.
- It is thus reasonable to assume that we have an infinite number of records: \( x_0 = \sum_{t=1}^{\infty} f_t \cdot x_t \).
- In practice, we only know values \( x_1, \ldots, x_T \).
- Thus, we use an approximate formula
  \[
  x_0 \approx \sum_{t=1}^{T} f_t \cdot x_t.
  \]
5. Case of a Constant Signal

• In some cases, the observed signal $x_t$ does not change at all: $x_t = c$.

• In this case, it is reasonable to predict the same value $x_0 = c$.

• In other words, if $x_1 = x_2 = \ldots = c$, then

$$x_0 = \sum_{t=1}^{\infty} f_t \cdot x_t = c.$$ 

• In precise terms, this means that $\sum_{t=1}^{\infty} f_t \cdot c = c$.

• In particular, for $c = 1$, we get $\sum_{t=1}^{\infty} f_t = 1$. 
6. Exponential Smoothing: a Brief Reminder

- *General formula:* \( x_0 = \sum_{t=1}^{\infty} f_t \cdot x_t. \)

- *Question:* which predictor is the best?

- *Empirical fact:* exponential smoothing is one of the best in econometrics: \( f_t = \alpha \cdot (1 - \alpha)^{t-1}. \)

- *It is widely used:* described in textbooks, used in serious econometric studies.

- *Why exponential smoothing?* there exist many explanations for the usefulness of exponential smoothing.

- *Remaining problem:* these explanations are based on complex, not very intuitively clear statistical models.

- *What we do:* we provide a new (and rather simple) theoretical explanation of exponential smoothing.
7. Definitions

- By a *time series* \( x \), we mean an infinite sequence of real numbers \( x_1, \ldots, x_n, \ldots \).

- By a *predictor function* \( f \), we mean an infinite sequence of real numbers \( f_1, \ldots, f_n, \ldots \) for which
  \[
  \sum_{t=1}^{\infty} f_t = 1.
  \]

- By the *prediction* \( X_0(f, x) \) made by the predictor function \( f_t \) for the time series \( x_t \), we mean the value
  \[
  \sum_{t=1}^{\infty} f_t \cdot x_t.
  \]

- By a *noise pattern* \( p \), we mean a finite sequence of real numbers \( p_1, \ldots, p_k \).
8. Definitions (cont-d)

- Let $c$ be a real number, and let $m$ be a natural number.
- By $x(p, c, m)$, we mean a time series for which $x_{m+i} = p_i$ for $i = 1, \ldots, k$, and $x_t = c$ for all other $t$.
- We say that this time series $x(p, c, m)$ corresponds to
  - a constant signal plus
  - a noise pattern $p$ before moment $m$.
- We say that for a predictor function $f_t$, the effect of noise always decreases with time if:
  - for every noise pattern $p$, for every real number $c$ and
  - for every two natural numbers $m > m'$,
we have

\[ |X_0(f, x(p, c, m)) - c| \leq |X_0(f, x(p, c, m')) - c|. \]
9. **Our First Result: A Simple Justification of Exponential Smoothing**

- **Result:**
  - For every $\alpha \in (0, 2)$, for $f_t = \alpha \cdot (1 - \alpha)^{t-1}$, the effect of noise always decreases with time.
  - If for a function $f_t$, the effect of noise always decreases with time, then there exists $\alpha \in (0, 2)$ s.t.:
    \[
    f_t = \alpha \cdot (1 - \alpha)^{t-1}.
    \]

- **Discussion:**
  - Exponential smoothing is the only predictor for which the effect of noise always decreases with time.
  - Thus, the need to satisfy this natural property explains the efficiency of exponential smoothing.
10. **Price Transmission: Reminder**

- The price of a manufacturing product is determined by the price of the components and the price of the labor.
- If one of the component prices changes, this change affects the product’s price.
- This change is called *price transmission*.
- Example:
  - when the oil price changes, the gasoline prices change as well;
  - when the gasoline prices change, the transportation prices change as well;
  - when the transportation prices change, the price of transported goods changes.
11. Asymmetric Price Transmission

- When the component (input) price increases, the final product (output) price starts increasing right away.
- On the other hand, when the input price starts decreasing back, the output price decreases much slower.
- As a result:
  - when the input price falls to the original lower level,
  - the output price remains much higher than the original one.
- This phenomenon seems to contradict to the usual economic assumption:
  - that markets are efficient, and
  - that the price of each product is determined by the equilibrium of supply and demand.
12. Asymmetric Price Transmission: A Problem

- **Good news:** there exist explanations of this phenomenon.
- **Problem:** these explanations are based on complex models and are far from intuitive clarity.
- **What we do:** provide a simple explanation for asymmetric price transfer.
- **Our explanation:** the output price is determined not by the current input price, but by the predicted price.
- **Example:** suppose that oil was $20/barrel, then shoots to $100, then goes down to $20 and stays at $20.
- **Predicted price:** for simplicity, \( x_0 = (x_1 + x_2)/2 \).
- First, it is \((20 + 20)/2 = 20\), then \((20 + 100)/2 = 60\), then \((100 + 20)/2 = 60\), and only then \((20 + 20)/2 = 20\).
- **Result:** a one year delay in price decrease.
13. Additional Intuitive Arguments in Favor of Our Explanation

• **Situation:**
  – the price of the component remains stable and
  – then experiences a sudden decrease.

• **Our prediction:** a sudden decrease in the customer price of a final product as well.

• **We indeed observe** such a phenomenon on the example of consumer electronics:
  – when the computer chips become cheaper,
  – many electronic products become cheaper as well.

• **In real life,** due to inflation, cases when consumer prices go down are rarer.
14. Second Example: Predicting Trend Reversal

- So far, we described the problem of predicting the new value when we are within a certain trend.
- Another important problem is predicting when a trend will reverse (e.g., when recession will end).
- It is a known empirical fact that volatility tends to increase before trend reversals.
- Thus, such volatility increases are a known predictor of trend reversals.
- This empirical fact is somewhat counter-intuitive:
  - when the trend changes from decrease to increase, the corresponding quantity reaches its minimum;
  - at the minimum, derivative is 0 – i.e., the local change is the smallest;
  - in economics, vice versa, the local change is the largest when trend reverses.
15. Towards an Explanation

• As an example of a time series, let us consider a stock price.

• With respect to the given stock, some traders are optimistic, some are pessimistic.

• An **optimistic** trader:
  – believes that the stock will rise,
  – so he/she is willing to pay a little extra for this stock,
  – in the expectation of larger gains in the future.

• A **pessimistic** trader:
  – believes that the stock will go down,
  – so he/she is willing to sell this stock even for a lower price than most.
16. Towards an Explanation (cont-d)

- The overall price of the stock can be computed as an average over all the transactions.
- Let $x$ be the last recorded price for the stop.
- Let $\delta$ be the average value of the small increase/decrease in stock in transactions by optimists and pessimists.
- We are interested in the average behavior of all the traders in the market.
- In such an average behavior, individual differences tend to average out.
- Thus, it seems safe to simply assume that:
  - each optimist performs transactions with this stock at the price $x + \delta$, while
  - each pessimist performs transactions at the price $x - \delta$. 
17. Towards an Explanation (cont-d)

• Let $p$ be the proportion of optimists, i.e., the probability that a randomly selected trader is an optimist.

• To further simplify our description, we will also assume that all the traders are independent from each other.

• Let $n$ denote the total number of traders. Thus, we arrive at the following model.

• We start with the price $x$.

• At the next moment of time, we have a price

$$x' = \frac{x_1 + \ldots + x_n}{n},$$

where:

• $x_i = x + \eta_i \cdot \delta$, and

• $\eta_i = \pm 1$, with $\text{Prob}(\eta_i = 1) = p$. 
18. Analysis of the Resulting Model

- **Reminder:** $x' = \frac{x_1 + \ldots + x_n}{n}$, where $x_i = x + \eta_i \cdot \delta$ and $\text{Prob}(\eta_i = 1) = p$.

- **Here:** $E[x'] = E[x_i] = x + (p \cdot 1 + (-1) \cdot (1 - p))$, so the mean price is $E[x'] = x + (2p - 1) \cdot \delta$.

- **Conclusion:** the price increases when $p > 1/2$ and decreases when $p < 1/2$.

- **Conclusion:** the trend reverses when $p = 1/2$.

- **Natural measure of volatility:** standard deviation $\sigma$.

- **Result of computations:** $\sigma = 2 \cdot \delta \cdot \frac{1}{\sqrt{n}} \cdot \sqrt{p \cdot (1 - p)}$.

- **Interesting:** the value $\sigma$ is the smallest when $p = 0$ and $p = 1$ and attains its largest value when $p = 1/2$.

- **Conclusion:** volatility is indeed the largest when the trend reverses – exactly as empirically observed.
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20. Appendix: Main Idea Behind the Proof of the Main Result

• **Main requirement:** the effect of noise pattern decreases with time.

• **Example:** two-value pattern \((p_1, p_2)\) with \(p_2 = 1\).

• **Requirement:** \(|f_{m+1} \cdot p_1 + f_{m+2}| \leq |f_m \cdot p_1 + f_{m+1}|\).

• **Hence:** for \(p_1 = -\frac{f_{m+1}}{f_m}\), we have \(f_m \cdot p_1 + f_{m+1} = 0\).

• **Conclusion:** we must have \(f_{m+1} \cdot p_1 + f_{m+2} = 0\).

• **Conclusion:** \(p_1 = -\frac{f_{m+1}}{f_m} = -\frac{f_{m+2}}{f_{m+1}}\).

• **By induction:** we get \(\frac{f_2}{f_1} = \frac{f_3}{f_2} = \ldots = \frac{f_{m+1}}{f_m} = \ldots\)

• **Thus:** \(f_t = f_1 \cdot (1 - \alpha)^{t-1}\), where \(\alpha = 1 + p_1\).

• **Here:** \(\sum_{t=1}^{\infty} f_t = 1\) leads to \(f_1 = \alpha\), so \(f_t = \alpha \cdot (1 - \alpha)^{t-1}\).