

# Continuous If-Then Statements Are Computable

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## 1. Outline

- *In many practical situations:* we must compute the value of an if-then expression

“if  $c(x) \geq 0$  then  $f_+(x)$  else  $f_-(x)$ ”.

- *Fact:* the value  $f(x)$  cannot be computed directly.
- *Reason:* in general, it is not possible to check whether a given real number  $c(x)$  is non-negative or non-positive.
- *Conclusion:* it is not possible to compute the value  $f(x)$  if the if-then function is discontinuous.
- *We show:* if the if-then expression is continuous, then we can effectively compute  $f(x)$ .

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## 2. Practical Need for If-Then Statements

- *In many practical situations:* we have different models for describing a phenomenon:
  - a model  $f_+(x)$  corresponding to the case when a certain constraint  $c(x) \geq 0$  is satisfied, and
  - a model  $f_-(x)$  corresponding to the case when this constraint is not satisfied, i.e., when  $c(x) < 0$ .
- *Comment:* usually, the second model is also applicable when  $c(x) \leq 0$ .
- *Example:* gravitational force generated by a celestial body:
  - outside the body, when  $c(x) \stackrel{\text{def}}{=}} \|\vec{r}\| - R \geq 0$ , we have  $f_+(x) = G \cdot m \cdot M \cdot r^{-2}$ ;
  - inside the body, when  $c(x) \leq 0$ , we have  $f_-(x) = G \cdot m \cdot M \cdot r \cdot R^{-3}$ .

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### 3. Towards a Precise Formulation of the Computational Problem

- We know how to compute  $f_+(x)$ ,  $f_-(x)$ , and  $c(x)$ .
- We want to compute the “if-then” function

$$f(x) \stackrel{\text{def}}{=} \text{if } c(x) \geq 0 \text{ then } f_+(x) \text{ else } f_-(x).$$

- $f(x)$  is *computable* if there is an algorithm that,
  - given an input  $x$  and a rational number  $\varepsilon > 0$ ,
  - produces a rational number  $r$  s.t.  $|f(x) - r| \leq \varepsilon$ .
- *Here:*
  - $c(x)$  is computable for all possible values  $x$  from a given set  $X$ ;
  - $f_+(x)$  is computable for all  $x \in X$  s.t.  $c(x) \geq 0$ ;
  - $f_-(x)$  is computable for all  $x \in X$  s.t.  $c(x) \leq 0$ .

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## 4. Why This Problem Is Non-Trivial

- *Reminder:* we want to compute the if-then function

$$f(x) = \text{“if } c(x) \geq 0 \text{ then } f_+(x) \text{ else } f_-(x)\text{”}.$$

- *Observation:*  $f(x)$  cannot be computed directly.
- *Proof:* in general, it is not possible to check whether a given real number  $c(x)$  is non-negative or non-positive.
- *Known result:* every computable function is everywhere continuous.
- *Conclusion:* when  $f_+(x_0) \neq f_-(x_0)$  for some  $x_0$  for which  $c(x_0) = 0$ , then  $f(x)$  is not computable.
- *Our main result:* in all other cases,  $f(x)$  is computable.
- *In other words:* when the if-then function  $f(x)$  is continuous, it is computable.

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## 5. Analysis: Main Idea and Case 1

- For every  $x$ , we have one of the three possible cases:
  1. case of  $c(x) > 0$ , when  $f(x) = f_+(x)$ ;
  2. case of  $c(x) < 0$ , when  $f(x) = f_-(x)$ ; and
  3. case of  $c(x) = 0$ , when  $f(x) = f_+(x) = f_-(x)$ .
- *Case 1:*  $c(x) > 0$ , when  $f(x) = f_+(x)$ .
  - Let us compute  $c(x)$  with higher and higher accuracy  $\varepsilon = 2^{-k}$ ,  $k = 1, 2, \dots$
  - Eventually, we reach the accuracy  $2^{-k} < \frac{c(x)}{2}$ , for which  $c(x) > 2 \cdot 2^{-k}$ .
  - Then, we get  $r_k$  s.t.  $|c(x) - r_k| \leq 2^{-k}$ , hence  $r_k > c(x) - 2^{-k} \geq 2 \cdot 2^{-k} - 2^{-k} = 2^{-k}$ , so  $r_k > 2^{-k}$ .
  - Since we know that  $c(x) \geq r_k - 2^{-k}$ , we thus conclude that  $c(x) > 0$ .

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## 6. Analysis: Case 2

- Case 2 (reminder):  $c(x) < 0$ , when  $f(x) = f_-(x)$ .
- Let us compute  $c(x)$  with higher and higher accuracy  $\varepsilon = 2^{-k}$ ,  $k = 1, 2, \dots$
- Eventually, we reach the accuracy  $2^{-k} < \frac{|c(x)|}{2}$ , for which  $c(x) < -2 \cdot 2^{-k}$ .
- Then, we get  $r_k$  s.t.  $|c(x) - r_k| \leq 2^{-k}$ , hence
$$r_k < c(x) + 2^{-k} \leq -2 \cdot 2^{-k} + 2^{-k} = -2^{-k}.$$
- So,  $r_k < -2^{-k}$ .
- Since we know that  $c(x) \leq r_k + 2^{-k}$ , we thus conclude that  $c(x) < 0$ .

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## 7. Analysis: Case 3

- Case 3:  $c(x) = 0$  and  $f(x) = f_-(x) = f_+(x)$ .
- Let us compute  $f_+(x)$  and  $f_-(x)$  with accuracy  $\varepsilon > 0$ , getting  $r_+$ ,  $r_-$ , and  $\bar{r} = \frac{r^- + r^+}{2}$ .

- Then,  $|f(x) - r_+| = |f_+(x) - r_+| \leq \varepsilon$  and  $|f(x) - r_-| = |f_-(x) - r_-| \leq \varepsilon$ , hence

$$|r_+ - r_-| \leq |r_+ - f(x)| + |f(x) - r_-| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

- Vice versa, let  $|r_+ - r_-| \leq 2\varepsilon$ .
- If  $f(x) = f_+(x)$ , then  $|f(x) - r_+| = |f_+(x) - r_+| \leq \varepsilon$  and

$$|f(x) - r_-| = |f_+(x) - r_-| \leq |f_+(x) - r_+| + |r_+ - r_-| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

- Thus, due to convexity of the absolute value, we have

$$|f(x) - \bar{r}| \leq \frac{1}{2} \cdot (|f(x) - r_+| + |f(x) - r_-|) \leq \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

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## 8. Analysis: Case 3 (cont-d)

- *Reminder:* let  $|r_+ - r_-| \leq 2\varepsilon$  and  $\bar{r} \stackrel{\text{def}}{=} \frac{r_- + r_+}{2}$ .
- *Already analyzed:* subcase when  $f(x) = f_+(x)$ .
- If  $f(x) = f_-(x)$ , then  $|f(x) - r_-| = |f_-(x) - r_-| \leq \varepsilon$  and

$$|f(x) - r_+| = |f_-(x) - r_+| \leq |f_-(x) - r_-| + |r_- - r_+| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

- Thus, due to convexity of the absolute value, we have

$$|f(x) - \bar{r}| \leq \frac{1}{2} \cdot (|f(x) - r_-| + |f(x) - r_+|) \leq \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

- *Summarizing:* in both case, we have  $|f(x) - \bar{r}| \leq 2\varepsilon$ .
- *We want:* compute  $f(x)$  with a given accuracy  $\alpha > 0$ .
- *Idea:* find  $\frac{\alpha}{2}$ -approximations  $r_-$  (to  $f_-(x)$ ) and  $r_+$  (to  $f_+(x)$ ) s.t.  $|r_+ - r_-| \leq \alpha$ , and take  $\bar{r} = \frac{r_- + r_+}{2}$ .

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## 9. Resulting Algorithm for Computing $f(x)$ with a Given Accuracy $\alpha$

- Simultaneously run the following three processes:
  1. computing  $c(x)$  with accuracy  $\varepsilon = 2^{-k}$ ,  $k = 1, 2, \dots$ ;
  2. computing  $f_-(x)$  with accuracy  $\frac{\alpha}{2}$ ;
  3. computing  $f_+(x)$  with accuracy  $\frac{\alpha}{2}$ .
- *Notations for results:* (1)  $r_k$ , (2)  $r_-$ , (3)  $r_+$ .
- *Above analysis:* eventually, one of the following occurs:
  - $r_k > 2^{-k}$ ; then,  $c(x) > 0$ , hence the third process will finish, so we finish it and return  $r_+$ ;
  - $r_k < -2^{-k}$ ; then,  $c(x) < 0$  hence the second process will finish, so we finish it and return  $r_-$ ;
  - $|r_+ - r_-| \leq \alpha$ ; in this case, we return  $\bar{r} = \frac{r_- + r_+}{2}$ .

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## 11. Historical Comment

Our proof is a simplified version of the proofs described, in a more general setting, in:

- Weihrauch, K.: *Computable Analysis: An Introduction*, Springer-Verlag, New York, 2000. see also Brattka et al.

see also

- Brattka, V., Gherardi, G.: Weihrauch degrees, omniscience principle, and weak computability, In: Bauer, A., Dillhage, R., Hertling, P., Ko K.-I, and Rettinger, R. (eds.), *Proceedings of the Sixth International Conference on Computability and Complexity in Analysis CCA'2009*, Ljubljana, Slovenia, August 18–22, 2009, pp. 81–92.

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