

Continuous If-Then Statements Are Computable

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1. Outline

- *In many practical situations:* we must compute the value of an if-then expression

“if $c(x) \geq 0$ then $f_+(x)$ else $f_-(x)$ ”.

- *Fact:* the value $f(x)$ cannot be computed directly.
- *Reason:* in general, it is not possible to check whether a given real number $c(x)$ is non-negative or non-positive.
- *Conclusion:* it is not possible to compute the value $f(x)$ if the if-then function is discontinuous.
- *We show:* if the if-then expression is continuous, then we can effectively compute $f(x)$.

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2. Practical Need for If-Then Statements

- *In many practical situations:* we have different models for describing a phenomenon:
 - a model $f_+(x)$ corresponding to the case when a certain constraint $c(x) \geq 0$ is satisfied, and
 - a model $f_-(x)$ corresponding to the case when this constraint is not satisfied, i.e., when $c(x) < 0$.
- *Comment:* usually, the second model is also applicable when $c(x) \leq 0$.
- *Example:* gravitational force generated by a celestial body:
 - outside the body, when $c(x) \stackrel{\text{def}}{=}} \|\vec{r}\| - R \geq 0$, we have $f_+(x) = G \cdot m \cdot M \cdot r^{-2}$;
 - inside the body, when $c(x) \leq 0$, we have $f_-(x) = G \cdot m \cdot M \cdot r \cdot R^{-3}$.

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3. Towards a Precise Formulation of the Computational Problem

- We know how to compute $f_+(x)$, $f_-(x)$, and $c(x)$.
- We want to compute the “if-then” function

$$f(x) \stackrel{\text{def}}{=} \text{if } c(x) \geq 0 \text{ then } f_+(x) \text{ else } f_-(x).$$

- $f(x)$ is *computable* if there is an algorithm that,
 - given an input x and a rational number $\varepsilon > 0$,
 - produces a rational number r s.t. $|f(x) - r| \leq \varepsilon$.
- *Here:*
 - $c(x)$ is computable for all possible values x from a given set X ;
 - $f_+(x)$ is computable for all $x \in X$ s.t. $c(x) \geq 0$;
 - $f_-(x)$ is computable for all $x \in X$ s.t. $c(x) \leq 0$.

4. Why This Problem Is Non-Trivial

- *Reminder:* we want to compute the if-then function

$$f(x) = \text{“if } c(x) \geq 0 \text{ then } f_+(x) \text{ else } f_-(x)\text{”}.$$

- *Observation:* $f(x)$ cannot be computed directly.
- *Proof:* in general, it is not possible to check whether a given real number $c(x)$ is non-negative or non-positive.
- *Known result:* every computable function is everywhere continuous.
- *Conclusion:* when $f_+(x_0) \neq f_-(x_0)$ for some x_0 for which $c(x_0) = 0$, then $f(x)$ is not computable.
- *Our main result:* in all other cases, $f(x)$ is computable.
- *In other words:* when the if-then function $f(x)$ is continuous, it is computable.

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5. Analysis: Main Idea and Case 1

- For every x , we have one of the three possible cases:
 1. case of $c(x) > 0$, when $f(x) = f_+(x)$;
 2. case of $c(x) < 0$, when $f(x) = f_-(x)$; and
 3. case of $c(x) = 0$, when $f(x) = f_+(x) = f_-(x)$.
- *Case 1:* $c(x) > 0$, when $f(x) = f_+(x)$.
 - Let us compute $c(x)$ with higher and higher accuracy $\varepsilon = 2^{-k}$, $k = 1, 2, \dots$
 - Eventually, we reach the accuracy $2^{-k} < \frac{c(x)}{2}$, for which $c(x) > 2 \cdot 2^{-k}$.
 - Then, we get r_k s.t. $|c(x) - r_k| \leq 2^{-k}$, hence $r_k > c(x) - 2^{-k} \geq 2 \cdot 2^{-k} - 2^{-k} = 2^{-k}$, so $r_k > 2^{-k}$.
 - Since we know that $c(x) \geq r_k - 2^{-k}$, we thus conclude that $c(x) > 0$.

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6. Analysis: Case 2

- *Case 2 (reminder):* $c(x) < 0$, when $f(x) = f_-(x)$.
- Let us compute $c(x)$ with higher and higher accuracy $\varepsilon = 2^{-k}$, $k = 1, 2, \dots$
- Eventually, we reach the accuracy $2^{-k} < \frac{|c(x)|}{2}$, for which $c(x) < -2 \cdot 2^{-k}$.
- Then, we get r_k s.t. $|c(x) - r_k| \leq 2^{-k}$, hence
$$r_k < c(x) + 2^{-k} \leq -2 \cdot 2^{-k} + 2^{-k} = -2^{-k}.$$
- So, $r_k < -2^{-k}$.
- Since we know that $c(x) \leq r_k + 2^{-k}$, we thus conclude that $c(x) < 0$.

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7. Analysis: Case 3

- Case 3: $c(x) = 0$ and $f(x) = f_-(x) = f_+(x)$.
- Let us compute $f_+(x)$ and $f_-(x)$ with accuracy $\varepsilon > 0$, getting r_+ , r_- , and $\bar{r} = \frac{r^- + r^+}{2}$.

- Then, $|f(x) - r_+| = |f_+(x) - r_+| \leq \varepsilon$ and $|f(x) - r_-| = |f_-(x) - r_-| \leq \varepsilon$, hence

$$|r_+ - r_-| \leq |r_+ - f(x)| + |f(x) - r_-| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

- Vice versa, let $|r_+ - r_-| \leq 2\varepsilon$.
- If $f(x) = f_+(x)$, then $|f(x) - r_+| = |f_+(x) - r_+| \leq \varepsilon$ and

$$|f(x) - r_-| = |f_+(x) - r_-| \leq |f_+(x) - r_+| + |r_+ - r_-| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

- Thus, due to convexity of the absolute value, we have

$$|f(x) - \bar{r}| \leq \frac{1}{2} \cdot (|f(x) - r_+| + |f(x) - r_-|) \leq \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

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8. Analysis: Case 3 (cont-d)

- *Reminder:* let $|r_+ - r_-| \leq 2\varepsilon$ and $\bar{r} \stackrel{\text{def}}{=} \frac{r_- + r_+}{2}$.
- *Already analyzed:* subcase when $f(x) = f_+(x)$.
- If $f(x) = f_-(x)$, then $|f(x) - r_-| = |f_-(x) - r_-| \leq \varepsilon$ and

$$|f(x) - r_+| = |f_-(x) - r_+| \leq |f_-(x) - r_-| + |r_- - r_+| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

- Thus, due to convexity of the absolute value, we have

$$|f(x) - \bar{r}| \leq \frac{1}{2} \cdot (|f(x) - r_-| + |f(x) - r_+|) \leq \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

- *Summarizing:* in both case, we have $|f(x) - \bar{r}| \leq 2\varepsilon$.
- *We want:* compute $f(x)$ with a given accuracy $\alpha > 0$.
- *Idea:* find $\frac{\alpha}{2}$ -approximations r_- (to $f_-(x)$) and r_+ (to $f_+(x)$) s.t. $|r_+ - r_-| \leq \alpha$, and take $\bar{r} = \frac{r_- + r_+}{2}$.

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9. Resulting Algorithm for Computing $f(x)$ with a Given Accuracy α

- Simultaneously run the following three processes:
 1. computing $c(x)$ with accuracy $\varepsilon = 2^{-k}$, $k = 1, 2, \dots$;
 2. computing $f_-(x)$ with accuracy $\frac{\alpha}{2}$;
 3. computing $f_+(x)$ with accuracy $\frac{\alpha}{2}$.
- *Notations for results:* (1) r_k , (2) r_- , (3) r_+ .
- *Above analysis:* eventually, one of the following occurs:
 - $r_k > 2^{-k}$; then, $c(x) > 0$, hence the third process will finish, so we finish it and return r_+ ;
 - $r_k < -2^{-k}$; then, $c(x) < 0$ hence the second process will finish, so we finish it and return r_- ;
 - $|r_+ - r_-| \leq \alpha$; in this case, we return $\bar{r} = \frac{r_- + r_+}{2}$.

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11. Historical Comment

Our proof is a simplified version of the proofs described, in a more general setting, in:

- Weihrauch, K.: *Computable Analysis: An Introduction*, Springer-Verlag, New York, 2000. see also Brattka et al.

see also

- Brattka, V., Gherardi, G.: Weihrauch degrees, omniscience principle, and weak computability, In: Bauer, A., Dillhage, R., Hertling, P., Ko K.-I, and Rettinger, R. (eds.), *Proceedings of the Sixth International Conference on Computability and Complexity in Analysis CCA'2009*, Ljubljana, Slovenia, August 18–22, 2009, pp. 81–92.

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