How to Faster Test a Device for Different Combinations of Parameters

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1. Formulation of the Problem

- Many devices have to function correctly under many different values of the corresponding parameters: e.g.,
  - for temperatures within the given range,
  - for pressure within the given range,
  - for humidity within the given range, etc.

- Ideally, we should test the device for all possible combinations of the corresponding parameters.

- However, often, such a testing is not realistic. For example:
  - if we have 20 possible parameters, and
  - we consider 10 possible values of each of these parameters,
  - then $10^{20}$ tests require $3 \cdot 10^{12}$ years – longer than the lifetime of the Universe.
2. Solution: Test for All Pairs, or All Triples, etc.

• We cannot test for all possible combinations of all the parameters.

• So, we need to test at least for all possible values of each parameter separately:
  – for all possible values of outside temperature,
  – for all possible values of humidity, etc.

• In this testing, we may overlook possible joint effect of two or more different parameters.

• To take such an effect into account, it makes sense to test all pairs of values.

• Similarly, we may want to test all possible triples of values, etc.
3. How to Arrange Such a Test: First Simple Idea

- Let us assume that for each of $n$ parameters, we test for $N$ different values.
- In this case, we need $n \cdot N$ experiments to test the device’s behavior for all $N$ values of each of $n$ parameters.
- For each of $\binom{n}{2}$ pairs of parameters, we test all possible $N^2$ pairs of values.
- Thus, we need $\binom{n}{2} \cdot N^2$ experiments.
- For each of $\binom{n}{k}$ $k$-tuples of parameters, we test all possible $N^k$ tuples of values.
- Thus, we need $\binom{n}{k} \cdot N^k$ experiments.
4. We Can Test Faster than That

- To test all possible values of each parameter, the above approach requires $n \cdot N$ experiments.

- In reality, it is sufficient to perform only $N$ experiments:
  - in the first experiment, we select the first value of each of $n$ parameters;
  - in the second experiment, we select the second value of each of $n$ parameters; etc.

- When we have many parameters $n \gg 1$, we then have $n \cdot N \gg N$.

- So, this idea drastically decreases the number of necessary experiments – and thus, the testing time.

- We show that a similar speed-up is possible when we test all possible pairs (triples, etc.) of parameters.
5. Formulating the Problem in Precise Terms

- Let \( n, N, \) and \( k \) be positive natural numbers.
- The number \( n \) will be called the number of parameters, and the number \( N \) will be called the number of values.
- By an experiment, we mean a tuple of \( n \) integers \( j_1, \ldots, j_n \), where \( 1 \leq j_i \leq N \) for all \( i \).
- By a testing design, we mean a finite set of experiments.
- We say that a testing design \( T \) tests each combination of \( k \) parameters if:
  - for every two \( k \)-tuples \( 1 \leq i_1 < \ldots < i_k \leq N \) and \( v_1, \ldots, v_k \) \( (1 \leq v_\ell \leq N) \),
  - \( T \) contains an experiment in which we use the \( v_\ell \)-th value of each \( i_\ell \)-th parameter.
6. Main Result

- **Objective:** minimize the number of experiments.

- **Naive idea:** tests each of \( \binom{n}{k} \cdot N^k = O(n^k) \cdot N^k \) combinations of \( k \) parameters.

- For \( n = k \), we need to test all \( N^k \) possible combinations of parameters, so we cannot have fewer than \( N^k \) tests.

- However, as the above case of \( k = 1 \) shows, we can try to minimize the factor depending on \( n \).

- For each \( k \), there is a design that tests each combination of \( k \) parameters in \( O(\log^{k-1}(n)) \cdot N^k \) experiments.

- For \( k = 1 \), we get the known fact that we need \( O(N) \) experiments.

- For testing all pairs \( (k = 2) \), we need \( O(\log(n)) \cdot N^2 \) experiments \( (\ll O(n^2) \cdot N^2 \) experiments in naive case).
7. **New Testing Design: Case** $k = 2$

- Our design consists of $B = \lceil \log_2(n) \rceil \sim \log(n)$ groups, each having $N^2$ experiments.
- Each number $x \leq n - 1$ can be represented by $B$ bits $\text{bit}_j(x)$, $j = 1, \ldots, B$.
- In the $b$-th group, for each pair of integers $(f, s)$ such that $1 \leq f, s \leq N$, we take:
  - $j_i = f$ if $\text{bit}_b(i - 1) = 0$, and
  - $j_i = s$ if $\text{bit}_b(i - 1) = 1$.
- For each pair $i_1 < i_2$, at least one bit $b$ in the binary expansions of $i_1 - 1$ and $i_2 - 1$ is different.
- For this bit $b$, the corresponding group of experiments tests all possible pairs $(f, s)$.
8. **First Example: \( n = 2 \)**

- For \( n = 2 \), we need \( B = 1 \) bit to represent integers 0 and 1.
- Thus, in this case, we have a single group of experiments: for each pair \((s, f)\), we set \( x_1 = f \) and \( x_2 = s \).
- In other words, each experiment has the form \((s, f)\).
9. Second Example: \( n = 4 \)

- For \( n = 4 \), we need \( B = 2 \) bits to represent 0, 1, 2, and 3: \( 0_{10} = 00_2, 1_{10} = 01_2, 2_{10} = 10_2, 3_{10} = 11_2 \).
- In this case, we have two groups of \( N^2 \) experiments.
- In the 1st group, we assign \( s \) to all \( i \) s.t. \( \text{bit}_1(i-1) = 0 \), and \( f \) to all \( i \) s.t. \( \text{bit}_1(i-1) = 1 \).
- Thus, each experiment has the form \((f, s, f, s)\).
- In the 2nd group, we assign \( s \) to all \( i \) s.t. \( \text{bit}_2(i-1) = 0 \), and \( f \) to all \( i \) s.t. \( \text{bit}_2(i-1) = 1 \).
- Thus, each experiment has the form \((f, f, s, s)\).
- If \( i_1 < i_2 \) are both odd or even, then the 2nd group of experiments tests all possible combinations of values.
- If one of \( i_1 \) and \( i_2 \) is odd and another even, then the 1st group tests all possible combinations of values.
10. Third Example: \( n = 8 \)

- For \( n = 8 \), we need \( B = 3 \) bits to represent integers from 0 to 7.

- Thus, in this case, we have three groups of \( N^2 \) experiments each.

- In the first group of experiments, each experiment has the form
  \[(f, s, f, s, f, s, f, s)\].

- In the second group of experiments, each experiment has the form
  \[(f, f, s, s, f, f, s, s)\].

- In the third group of experiments, each experiment has the form
  \[(f, f, f, f, s, s, s, s)\].
11. Testing Design: Case of $k > 2$

- For $n = k$, we just have to test all $N^k$ possible combinations of values of all $k$ parameters.
- For $n > k$, we divide the set of $n$ parameters into two halves of size $n/2$.
- To cover situations when all $k$ parameters are in the 1st half, we use the testing design for $n/2$ and $k$.
- Each experiment in this design is copied for the second half, so, e.g., a design $f s$ becomes $f s f s$.
- To cover situations in which $i$ parameters are in the 1st half, we combine:
  - each experiment from design for $n/2$ and $i$ with
  - each experiment from design for $n/2$ and $k - i$. 
12. Case of $k > 2$: Number of Experiments

- Let $E_k(n)$ be the number of experiments that our algorithm requires for $n$ and $k$; then:

$$E_k(n) = E_k(n/2) + \sum_{i=1}^{k-1} E_{k-i}(n/2) \cdot E_i(n/2).$$

- One can prove, by induction, that this implies that

$$E_k(n) = O(\log^k_2(n)) \cdot N^k.$$
13. **Example: \( n = 4 \) and \( k = 3 \)**

- It is not possible to have \( n/2 = 2 \) parameters and test all possible values of \( k = 3 \) of them.
- We combine each experiment with \( n = 2 \) and \( k = 2 \) with each experiment with \( n = 2 \) and \( k = 1 \).
- There is one group of experiments with \( n = 2 \) and \( k = 2 \): \( sf \), with \( s \) and \( f \) going from 1 to \( N \).
- There is one group corresponding to \( n = 2 \) and \( k = 1 \): \( tt \), with \( t \) from 1 to \( N \).
- Thus, by combining them, we get experiments of the type \( sftt \).
- Finally, we combine each experiment with \( n = 2 \) and \( k = 1 \) with each experiment with \( n = 2 \) and \( k = 2 \).
- Similarly, we get \( ttsf \).
- So, we get two groups of experiments: \( sftt \) and \( ttsf \).
14. Example: $n = 8$ and $k = 3$

- First, we list all the experiments corresponding to $n/2 = 4$ and $k = 3$, and repeat each for the second half as well.
- Thus, from $sftt$ and $ttsf$, we get $sftstftt$ and $tstftttsf$.
- Then, we combine each experiment with $n = 4$ and $k = 2$ with each experiment with $n = 4$ and $k = 1$.
- There are 2 groups of experiments with $n = 4$ and $k = 2$: $sfsf$ and $ssff$.
- There is one group for $n = 4$ and $k = 1$: $tttt$.
- Combining, we get $sfsttttt$ and $ssfftttt$.
- Finally, we combine each experiment with $n = 4$ and $k = 1$ with each experiment with $n = 4$ and $k = 2$.
- Combining, we get $tttstsf$ and $tttssff$.
- Totally, we have 6 groups of $N^3$ experiments.