For Quantum and Reversible Computing, Intervals Are More Appropriate Than General Sets, And Fuzzy Numbers Than General Fuzzy Sets

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1. Need for Quantum Computing

- Our current computers are very fast in comparison with what was available a few years ago.
- However, there are still computational tasks that necessitate even faster computers.
- To speed up computers, we need to squeeze in more cells and into the same volume.
- For that, we need to make cells as small as possible.
- Already, the existing cells contain a small number of molecules.
- If we decrease them further, they will contain a few molecules.
- Thus, we will need to take into account quantum effects.
2. Quantum Computing: Additional Advantages

- There are innovative algorithms specifically designed for quantum computing.
- We can decrease the time needed to find an element in an unsorted array of size $n$ from $n$ to $\sqrt{n}$ steps.
- We can reduce the time needed to factor large integers of $n$ digits from exponential to polynomial in $n$.
- This task is needed to decode currently encoded messages.
3. Need for Reversible Computing

- One challenge in designing quantum computers is that on the quantum level, all equations are time-reversible.
- In the traditional algorithms, even the simplest “and”-operation \( a, b \rightarrow a \& b \) is not reversible:
  - if we know its result \( a \& b = 0 \) = “false”,
  - we cannot uniquely reconstruct the input \((a, b)\).
- Reversibility is also important because, according to statistical physics:
  - any irreversible process means increasing entropy,
  - and this leads to heat emission.
- Overheating is one of the reasons why we cannot pack too many processing units into the same volume.
- So, to pack more, it is desirable to reduce this heat emission – e.g., by using only reversible computations.
4. Need to Take Uncertainty into Account

- We use computers mostly to process data.
- When processing data, we need to take into account that data comes from measurements.
- Measurements are never absolutely accurate.
- The measurement result $\tilde{x}$ is, in general, different from the actual value $x$ of the corresponding quantity.
- It is therefore necessary to take this uncertainty into account when processing data.
5. Need for Interval Uncertainty

• In many real life situations:
  – the only information that we have about the measurement error $\Delta x \overset{\text{def}}{=} \tilde{x} - x$ is
  – the upper bound $\Delta$ on its absolute value:
    $$|\Delta x| \leq \Delta.$$  

• Once we have a measurement result $\tilde{x}$, then:
  – the only information that we can conclude about the actual value $x$ is that
  – this value is somewhere in the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$.

• Such interval uncertainty indeed appears in many practical applications.
6. Data Processing under Interval Uncertainty

- In a data processing algorithm:
  - we take several inputs $x_1, \ldots, x_n$, and
  - we apply an appropriate algorithm to generate the result $y$ depending on these inputs.

- Let us denote this dependence by $f(x_1, \ldots, x_n)$.

- For each input $i$, we only know the interval $X_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ of possible values of $x_i$.

- Then, the only information that we can have about $y$ is that $y$ belongs to the set
  \[ Y = f(X_1, \ldots, X_n) \overset{\text{def}}{=} \{ f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n \}. \]

- When the sets $X_i$ are intervals and the function $f(x_1, \ldots, x_n)$ is continuous, the resulting set $Y$ is also an interval.
7. Interval Uncertainty (cont-d)

- In most practical situations, the measurement errors are relatively small.
- So, we can expand the function $f(x_1, \ldots, x_n)$ in Taylor series and retain only linear terms.
- Then, we get

$$f(x_1, \ldots, x_n) = f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n) \approx \tilde{y} - \sum_{i=1}^{n} c_i \cdot \Delta x_i, \quad \tilde{y} \overset{\text{def}}{=} f(\tilde{x}_1, \ldots, \tilde{x}_n), \quad c_i \overset{\text{def}}{=} \frac{\partial f}{\partial x_i} |_{x_i = \tilde{x}_i}.$$

- In other words, $f(x_1, \ldots, x_n)$ becomes a linear function:

$$f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i \cdot x_i, \quad c_0 \overset{\text{def}}{=} \tilde{y} - \sum_{i=1}^{n} c_i \cdot \tilde{x}_i.$$

- In other words, data processing can be, in effect, reduced to multiplication by a constant $c_i$ and addition.
8. When Is This Data Processing Reversible?

- Multiplication by a constant is always reversible.
- Indeed, if we know the interval $Y = c \cdot X$, then, we can reconstruct $X$ as $X = c^{-1} \cdot Y$.
- Addition $y = x_1 + x_2$ is also reversible.
- Indeed, if we know that $x_1 \in [x_1, \bar{x}_1]$ and $x_2 \in [x_1, \bar{x}_2]$, then $Y = [y, \bar{y}]$ has the form
  \[ Y = [x_1 + x_2, \bar{x}_1 + \bar{x}_2]. \]
- If we know $Y = [y, \bar{y}]$ and $X_1 = [x_1, \bar{x}_1]$, then we can reconstruct $X_2 = [x_2, \bar{x}_2]$ as
  \[ \bar{x}_2 = y - x_1 \text{ and } \bar{x}_2 = \bar{y} - \bar{x}_1. \]
9. From Interval Uncertainty to a More General Set Uncertainty

- In some cases:
  - in addition to knowing that values of $x$ are within a certain interval $[\underline{x}, \bar{x}]$,
  - we also know that some values from this interval are not possible.

- In this case, the set $X$ of possible values of $x$ is different from an interval.

- No matter how crude the measurements are, there is always an upper bound $\Delta$ on the measurement error.

- Thus, all possible values of $x$ are in the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$.

- Thus, the set $X$ is bounded.
10. Set Uncertainty (cont-d)

• In general, we can safely assume that the set \( X \) is closed.

• Indeed, suppose that \( x_0 \) is a limit point of the set.

• Then, for every \( \varepsilon > 0 \), there are elements \( x \in X \) is any \( \varepsilon \)-neighborhood \((x_0 - \varepsilon, x_0 + \varepsilon)\) of this value \( x_0 \).

• This means that:
  
  – no matter how accurately we measure the corresponding value,
  
  – we will not be able to distinguish between the limit value \( x_0 \) and a sufficient close value \( x \in X \).

• It is therefore reasonable to simply assume that \( x_0 \) is possible.

• Thus, we conclude that the set of possible values of \( x \) contains all its limit points, i.e., is closed.
11. Data Processing under Set Uncertainty

- Assume that we know the set $X_1$ of possible values of $x_1$, and we know the set $X_2$ of possible values of $x_2$.

- Then the set $Y \overset{\text{def}}{=} X_1 + X_2$ of possible values of the sum $y = x_1 + x_2$ is equal to

$$Y = \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$  

- If we add any non-interval bounded closed set $S$ to the class of all intervals, additions stops being reversible.

- For $\underline{S} \overset{\text{def}}{=} \inf\{x : x \in S\}$ and $\overline{S} \overset{\text{def}}{=} \sup\{x : x \in S\}$, we have

$$[\underline{S}, \overline{S}] + [\underline{S}, \overline{S}] = [\underline{S}, \overline{S}] + S(= [2\underline{S}, 2\overline{S}]).$$

- However, $[\underline{S}, \overline{S}] \neq S$. 

12. Case of Fuzzy Uncertainty

- In many real-life situations:
  - in addition to the guaranteed upper bound $\Delta$ on the absolute value of the measurement error,
  - with some degree of certainty $\beta$, measurement errors can be bounded by a smaller bound $\Delta(\beta) < \Delta$.

- As a result:
  - in addition to the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$ that is guaranteed to contain $x$ with 100% confidence,
  - we have several narrower intervals $[\tilde{x} - \Delta(\beta), \tilde{x} + \Delta(\beta)]$ that contain $x$ with confidence $\beta$.

- In other words, we have a nested family of intervals corresponding to different values $\beta$.

- The larger the $\beta$ (i.e., the higher the desired confidence), the wider the interval.
13. Case of Fuzzy Uncertainty (cont-d)

- Such a family of nested interval is, in effect, an equivalent way of representing a fuzzy number.

- If instead of intervals, we have more general sets $S(\beta)$, then we have a *fuzzy set*.

- The sets $S(\beta)$ are known as *\(\alpha\)-cuts* of the fuzzy set, where $\alpha \overset{\text{def}}{=} 1 - \beta$.

- For such fuzzy sets, we can define operations layer-by-layer:
  - for each $\beta$ (i.e., equivalently, for each $\alpha$),
  - we process all the sets (or intervals) corresponding to this value $\beta$. 
14. Case of Fuzzy Uncertainty (cont-d)

- Fuzzy numbers correspond to intervals, and general fuzzy sets to general sets.
- So, we conclude that addition is only reversible for fuzzy numbers.
- If we add any fuzzy set which is not a fuzzy number to fuzzy numbers, addition stops being reversible.
15. Intervals are Ubiquitous

- We showed that intervals (and fuzzy numbers) are preferable: they lead to reversible data processing.
- Interestingly, intervals (and fuzzy numbers) are indeed ubiquitous.
- They occur much much more frequently in practice as descriptions of uncertainty than any other sets.
- Why is that?
16. A Possible Explanation: Main Idea

- Let us recall why normal (Gaussian) distributions are ubiquitous.
- The usual explanation is that usually, there are many different independent sources of measurement error.
- As a result, the measurement error is a sum of a large number of small independent random variables.
- In the limit, when the number of terms increases, the distribution of the sum tends to normal.
- This is known as the Central Limit Theorem.
- This means that when the number of components is large, the corresponding distribution is close to normal.
- Thus, from the practical viewpoint, we can safely consider the distribution to be normal.
- In non-probabilistic case, the situation is similar.
17. Main Idea (cont-d)

- The measurement error is the sum of a large number $n$ of small independent error components:

$$\Delta x = \Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)}.$$  

- Let us assume that for each of the components $\Delta x^{(k)}$, we know the set $X^{(k)}$ of possible values.

- Then the set $S$ of possible values of their sum is equal to the sum of these sets:

$$X = X^{(1)} + \ldots + X^{(n)} = \{ \Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)} : \Delta x^{(1)} \in X^{(1)}, \ldots, \Delta x^{(n)} \in X^{(n)} \}.$$  

- It can be shown that, when $n$ increases, the resulting set $X$ also tends to an interval.
18. Need for a More Detailed Explanation

• The limit closeness is good.

• However, in practice, it is desirable to know exactly how close is the resulting set $X$ to an interval.

• For every positive real number $\varepsilon > 0$, two points $a$ and $b$ are $\varepsilon$-close is $|a - b| \leq \varepsilon$.

• It is therefore reasonable to say that the sets $A$ and $B$ are $\varepsilon$-close if:
  
  - every point $a \in A$ is $\varepsilon$-close to some point $b \in B$, and
  
  - every point $b \in B$ is $\varepsilon$-close to some point $a \in A$.

• The smallest value $\varepsilon$ with this property is known as the Hausdorff distance $d_H(A, B)$ between the two sets.
19. How to Measure Smallness of a Set

- The size of a set $A$ can be naturally measured by its diameter $\text{diam}(A)$.

- The diameter is the largest possible distance $d(a, a')$ between the two points $a, a'$ from this set.

- For bounded closed subsets $A$ of a real line, the diameter is equal to $\text{diam}(A) = \sup A - \inf A$. 


20. **Our Main Result**

- *If* \( \text{diam}(A_i) \leq \varepsilon \) *for all* \( i = 1, \ldots, n \), *then for* \( A = A_1 + \ldots + A_n \) *and for some interval* \( I \):

\[
d_H(A, I) \leq \varepsilon/2.
\]

- This bound cannot be improved, as shown by the following auxiliary result.

- *For every* \( n \), *there exist closed bounded sets* \( A_1, \ldots, A_n \) *for which* \( \text{diam}(A_i) \leq \varepsilon \) *for all* \( i \), *and for which*

\[
d_H(A, I) \geq \varepsilon/2 \text{ for all } I.
\]
21. Proof of the Main Result

- Let us show that the desired inequality holds from the interval \([a, \bar{a}]\), where:
  
  \[ a \overset{\text{def}}{=} a_1 + \ldots + a_n, \text{ where } a_i \overset{\text{def}}{=} \inf A_i, \text{ and } \]
  \[ \bar{a} \overset{\text{def}}{=} \bar{a}_1 + \ldots + \bar{a}_n, \text{ where } \bar{a}_i \overset{\text{def}}{=} \sup A_i. \]

- To prove the desired inequality, we need to show that:
  
  - every point \( a \in A \) is \((\varepsilon/2)\)-close to some point from the interval \( I = [a, \bar{a}] \), and
  
  - vice versa, that every point \( b \) from the interval \( I = [a, \bar{a}] \) is \((\varepsilon/2)\)-close to some point from the sum \( A \).

- Let us first prove that every point \( a \in A \) is \((\varepsilon/2)\)-close to some point from the interval \( I = [a, \bar{a}] \).

- Indeed, by definition of \( A \), every point \( a \in A \) has the form \( a = a_1 + \ldots + a_n, \text{ where } a_i \in A_i \) for all \( i \).
22. Proof of the Main Result (cont-d)

• Every point \( a_i \in A_i \) is bounded by this set’s inf and sup: \( a_i = \inf A_i \leq a_i \leq \sup A_i \leq \bar{a}_i \).

• Let us add up \( n \) such inequalities, and take into account that:
  
  • \( a = a_1 + \ldots + a_n \),
  
  • \( a = a_1 + \ldots + a_n \), and
  
  • \( \bar{a} = \bar{a}_1 + \ldots + \bar{a}_n \).

• We can then conclude that \( a \leq a \leq \bar{a} \), i.e., that the value \( a \) actually itself belongs to the interval \( I \).

• So, we can take \( b = a \), and get \( |a - b| = 0 \leq \varepsilon/2 \).

• Let us prove that, vice versa, every point \( b \) from the interval \( I \) is \( (\varepsilon/2) \)-close to some point \( a \in A \).

• Indeed, since all \( A_i \) are closed sets, they contain their limit points \( a_i = \inf A_i \in A_i \).
23. Proof of the Main Result (cont-d)

- Thus, $a = a_1 + \ldots + a_n \in A$.
- Since $b \in I$, we have $b \geq a$, so $b$ is larger than or equal to some point $a \in A$.
- Let us define $a_0 = \sup\{a \in A : a \leq b\}$.
- Since all $A_i$ are closed sets, the sum $A$ of these sets is also closed.
- So, $a_0$, as a limit of elements from $A$, also belongs to $A$.
- In the limit, from $a \leq b$, we conclude that $a_0 \leq b$.
- If $a_0 = \bar{a}$, then, from the fact that $a_0 \leq b \leq \bar{a}$, we conclude that $b = a_0 = \bar{a}$ and thus, $|a_0 - b| = 0 \leq \varepsilon/2$.
- Let us now consider the remaining case when

$$a_0 < \bar{a} = \bar{a}_1 + \ldots + \bar{a}_n.$$
24. Proof of the Main Result (cont-d)

• Since the point \( a_0 \) is in \( A \), it means that
  \[ a_0 = a_1 + \ldots + a_n \]
  for some \( a_i \in A_i \).

• For each \( i \), we have \( a_i \leq \sup A_i = \bar{a}_i \).

• The inequality \( a_0 < \bar{a} \) implies that we cannot have
  \( a_i = \bar{a}_i \) for all \( i \): otherwise, we would have
  \[ a_0 = a_1 + \ldots + a_n = \bar{a}_1 + \ldots + \bar{a}_n = \bar{a}. \]

• Thus, there exists an \( i \) for which \( a_i < \bar{a}_i \).

• Let us denote one such index by \( i_0 \); then \( a_{i_0} < \bar{a}_{i_0} \).

• Let us now consider a new point \( \bar{a}_0 \in A \) in forming
  which we replace \( a_{i_0} \) with \( \bar{a}_{i_0} \):
  \[ \bar{a}_0 = a_1 + \ldots + a_{i_0-1} + \bar{a}_{i_0} + a_{i_0+1} + \ldots + a_n. \]

• Here, we have \( \bar{a}_0 - a_0 = \bar{a}_{i_0} - a_{i_0} \).
25. Proof of the Main Result (cont-d)

• Thus, by the definition of the diameter, this difference is smaller than or equal to the diameter \( \text{diam}(A_{i_0}) \).

• This diameter is \( \leq \varepsilon \); thus, \( |\bar{a}_0 - a_0| \leq \varepsilon \).

• Since \( a_0 \) is the largest point from \( A \) which is \( \leq b \), and \( \bar{a}_0 > a_0 \), we conclude that \( a_0 \not\leq b \), i.e., that \( b < \bar{a}_0 \).

• So, we have \( a_0 \leq b < \bar{a}_0 \).

• The sum of the distances \( |b - a_0| \) and \( |b - \bar{a}_0| \) is equal to \( |\bar{a}_0 - a_0| \) and is, thus, smaller than or equal to \( \varepsilon \):

\[
|b - a_0| + |b - \bar{a}_0| \leq \varepsilon.
\]

• So, at least one of these distances must be \( \leq \varepsilon/2 \) (if they were both \( > \varepsilon/2 \), their sum would be \( > \varepsilon \)).

• In each of these two cases, we have a point from \( A \) (\( a_0 \) or \( \bar{a}_0 \)) which is \( (\varepsilon/2) \)-close to \( b \in I \). Q.E.D.
26. Proof of Auxiliary Result

- Let us take \( A_1 = \ldots = A_n = \{0, \varepsilon\} \).
- Then, as one can easily see,
  \[ A = A_1 + \ldots + A_n = \{0, \varepsilon, 2 \cdot \varepsilon, \ldots, n \cdot \varepsilon\}. \]
- Let us show, by reduction to a contradiction, that we cannot have \( d_H(A, I) < \varepsilon/2 \) for any interval \( I \).
- Indeed, suppose that such an interval exists.
- Then, by definition of the Hausdorff distance, for the point \( 0 \in A \), there exists a point \( b_1 \in I \) for which
  \[ |b_1 - 0| = |b_1| \leq d_H(A, I). \]
- Then, since \( b_1 \leq |b_1| \), we have \( b_1 \leq d_H(A, I) \).
- Since \( d_H(A, I) < \varepsilon/2 \), we thus have \( b_1 < \varepsilon/2 \).
- Similarly, for the point \( \varepsilon \in A \), there exists a point \( b_2 \in I \) for which \( |\varepsilon - b_2| \leq d_H(A, I) \).
27. Proof of Auxiliary Result (cont-d)

- Thus, \( \varepsilon - b_2 \leq d_H(A, I) \) and \( \varepsilon - d_H(A, I) \leq b_2 \).
- Since \( d_H(A, I) < \varepsilon/2 \), we thus have \( b_2 > \varepsilon - \varepsilon/2 = \varepsilon/2 \).
- Since \( I \) contains two points \( b_1 < \varepsilon/2 \) and \( b_2 > \varepsilon/2 \), it contains all the points in between, including \( b = \varepsilon/2 \).
- However, for this point \( b \in I \), the closest points from \( A \) are the points 0 and \( \varepsilon \).
- For both of them, the distance to \( b = \varepsilon/2 \) is equal to \( \varepsilon/2 \) and is, thus, larger than \( d_H(A, I) \).
- This contradicts to the definition of Hausdorff distance.
- Indeed, by this definition, every \( b \in I \) is \( d_H(A, I) \)-close to some point from \( A \).
- This contradiction proves that the inequality \( d_H(A, I) < \varepsilon/2 \) is impossible. So, \( d_H(A, I) \geq \varepsilon/2 \). Q.E.D.
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