Mathematical and Computational Aspects of a Joint Inversion Paper by M. Moorkamp, A. G. Jones, and S. Fishwick

Anibal Sosa and Vladik Kreinovich

Cyber-ShARE Center
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
usosaaguirre@miners.utep.edu, vladik@utep.edu
1. Formulation of the Geophysical Problem

• **Problem**: we are interested in some quantity \( q \).

• **Example**: we are interested in how the density \( \rho \) depends on the depth \( d \): \( \rho = \rho(d) \).

• **Situation**: we have several types \( t \) of measurement results \( t \), e.g., they use seismic data, resistivity, etc.

• **Measurement results**: for each type of data \( t \), we have measurement results \( m_{t,i}, i = 1, \ldots, n_t \).

• **Measurement accuracy**: for each measurement, we have estimates \( \sigma_{t,i} \) of the accuracy of this measurement.

• **Problem**: sometimes, we only have a general accuracy estimate \( \sigma_t \) for all measurements of type \( t \).

• **Solution**: in this case, we take \( \sigma_{t,i} \approx \sigma_t \).
2. Formulation of the Geophysical Problem (cont-d)

• **Reminder:**
  
  – we are interested in a quantity $q$;
  – we have measurement results $m_{t,i}$ of different types $t$;
  – we know (approximately) the accuracies $\sigma_{t,i}$ of different measurements.

• **Forward models** $M_t$ enables us, given $q$, to predict the corresponding measured values

\[ m_{i,t} \approx M_t(i, q). \]

• **Least Squares formulation:** find $q$ that minimizes

\[ \sum_t \Phi_t, \text{ where } \Phi_t \overset{\text{def}}{=} \sum_{i=1}^{n_t} \frac{(m_{t,i} - M_t(i, q))^2}{\sigma_{t,i}^2}. \]

• **Problem:** the accuracies $\sigma_{t,i}$ are only approximately known.
3. **Main Idea of the Paper**

- *Ideal case:* if we knew the exact accuracies $\sigma_{t,i}$, we could apply the Least Squares approach.
- *In practice:* we only know approximate values of $\sigma_{t,i}$.
- *Reason:* for some $t$, we systematically overestimate the measurement errors; for other $t$, we underestimate.
- Whether we over- or under-estimate depends on $t$.
- *Natural idea:* assume that the actual accuracies are $\sigma_{t,i}^{\text{act}} = k_t \cdot \sigma_{t,i}$.
- *Resulting solution:* for all possible combinations of the correction coefficients $k_t$, find $q$ that minimizes

  $$\sum_t \frac{1}{k_t^2} \cdot \Phi_t, \quad \text{where } \Phi_t = \sum_{i=1}^{n_t} \frac{(m_{t,i} - M_t(i, q))^2}{\sigma_{t,i}^2}.$$ 

- *Selection* of an appropriate solution ("model") $q$ is made by a geophysicist.
4. Pareto Optimality

• **Reminder:** for all possible combinations of the correction coefficients $k_t$, find $q$ that minimizes

$$\sum_t \frac{1}{k_t^2} \cdot \Phi_t, \text{ where } \Phi_t = \sum_{i=1}^{n_t} \frac{(m_{t,i} - M_t(i, q))^2}{\sigma_{t,i}^2}.$$  

• **Known:** this is $\Leftrightarrow$ finding all Pareto optimal solutions $q$, i.e., $q$ which are not worse than any other $q'$:

$q$ worse than $q' \Leftrightarrow (\Phi_t(q) \leq \Phi_t(q'))$ for all $t$ and

$$\Phi_t(q) < \Phi_t(q')$$ for some $t$.

• **How they find it:** use genetic algorithm, with the minimized function $O(q) \overset{\text{def}}{=} \#\{q' : q' \text{ worse than } q\}$. 

5. Genetic Algorithm: Brief Description

- Each \( q \) is a sequence of values: e.g., \( \rho(d_i) \) at different depths \( d_i \).
- We start with several randomly generated sequences.
- At each step, we repeatedly
  - select two sequences \( s_1 \) and \( s_2 \) – the smaller \( O(q) \), the larger probability of selection;
  - select random splitting locations, so \( s_i = s_{i1}s_{i2}s_{i3} \ldots \), where \( s_{i1} \) is before the 1st location, etc.;
  - combine \( s_1 \) and \( s_2 \) into a new sequence \( s_{11}s_{22}s_{13}s_{24} \ldots \);
  - mutate, i.e., randomly change some elements of the new sequence.
- These new sequences form a new generation, with which we deal on the next step.
- We repeat this procedure many (\( N \gg 1 \)) times.
6. Selecting a Single Model

- **Reminder:** we find all the solutions which are Pareto-optimal with respect to $\Phi = (\Phi_1, \Phi_2, \ldots)$.

- **Interesting case:** when we have two types of measurements.

- **In this case:** we find all the solutions which are Pareto-optimal with respect to $\Phi = (\Phi_1, \Phi_2)$.

- **Empirical fact:**
  - if we plot the dependence of $\ln(\Phi_1)$ on $\ln(\Phi_2)$, then
  - at the geophysically most meaningful solution, the corresponding curve has the largest curvature.

- **Name:** the corresponding curve is called an *L-curve*, since it has a sharp corner – like a letter L.

- **Resulting idea:** look for the solution at which the curvature is the largest.