Data Processing in the Presence of Interval Uncertainty and Erroneous Measurements: Practical Problems, Results, Challenges

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1. Formulation of the Problem

- There are *two main reasons* why measurement results differ from the actual values of the measured quantities:

- There is a *small* difference caused by the inaccuracy of the measuring instrument.

- This inaccuracy is characterized by *probabilistic* or *interval* uncertainty.

- Sometimes, due to an instrument malfunction or a human error, we get an erroneous measurement result (outlier) that is *drastically* different from the actual value.

- This uncertainty is usually characterized by a *proportion* of measurement results that could be erroneous (e.g., ≤ 1%).

- *Situation:* most data processing algorithms based on interval computations only take into account the first type of uncertainty.

- *Problem:* take the presence of erroneous measurements into account as well.
2. Sometimes, It Is Relatively Easy to Detect Outliers

- In some cases, when the data is smooth, we can (rather easily) detect the corresponding outliers.

- *Traditional engineering approach:* a new measurement result $x$ is classified as an outlier if $x \not\in [L, U]$, where

$$ L \overset{\text{def}}{=} E - k_0 \cdot \sigma, \quad U \overset{\text{def}}{=} E + k_0 \cdot \sigma, $$

and $k_0 > 1$ is pre-selected (most frequently, $k_0 = 2, 3,$ or $6$).

- *Minor problem:* in some practical situations, we only have intervals $x_i = [x_i^L, x_i^U]$.

- For different values $x_i \in x_i$, we get different $k_0$-sigma intervals $[L, U]$.

- A value $x$ is a guaranteed outlier if $x \not\in [L, U]$.

- *Conclusion:* to detect outliers, we must know the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.

- *Good news:* there exist algorithm for computing these ranges.

- *Not so good news:* in many practical situations, e.g., in non-destructive testing (NDT) of aeroplanes and roads, and in geophysical analysis, we are actually interested in unusual non-smooth data points.

- *Problem:* separating correct but unusual measurement results from the erroneous measurement results is a challenge.
3. Presence of Erroneous Measurements Make Problems Computationally Difficult

- **Known fact:** the presence of outliers turns easy-to-solve interval problems into difficult-to-solve (NP-hard) ones.

- **New result:** this difficulty may appear even without interval uncertainty.

- **Situation:** we know how the measured quantity \( y \) is related to the desired parameters \( x_j \).

- **Simplest case:** linear dependence, i.e., \( \sum_{j=1}^{n} a_{ij} \cdot x_j = y_i \), where \( y_i \) is the result of \( i \)-th measurement, and \( a_{ij} \) are (known) parameters corresponding to \( i \)-th measurement.

- **Problem:** given \( a_{ij} \), \( y_i \), and \( \varepsilon \in (0, 1) \), and constraints

\[
\sum_{j=1}^{n} a_{ij} \cdot x_j = y_i, \quad i = 1, \ldots, N
\]

check whether we can select a consistent set of \( N \cdot (1 - \varepsilon) \) constraints.
4. Result: The Problem Is NP-hard Even for the Linear Case

- **Idea**: reduce to a known NP-hard problem.

- **Subset sum**: given positive integers \(s_1, \ldots, s_n\), and \(s\), check whether \(s = \sum_{i=1}^{n} x_i \cdot s_i\) for some \(x_i \in \{0, 1\}\).

- **Reduction**: \(N = n/\varepsilon\) constraints:
  
  - 2\(n\) constraints \(x_1 = 0, x_1 = 1 \ldots, x_n = 0, x_n = 1\);
  
  - \(N - 2n\) identical constraints \(\sum s_i \cdot x_i = s\).

- Since \(0 \neq 1\), at most \(N - n\) are satisfied.

- If the subset problem has a solution, then:
  
  - all \(N - 2n\) constraints \(\sum s_i \cdot x_i = s\) are satisfied,
  
  - and for each \(i\), \(x_i = 0\) or \(x_i = 1\),

  to the total of \(N - n = N \cdot (1 - \varepsilon)\).

- If \(N - n\) constraints are satisfied, then for every \(i\), \(x_i \in \{0, 1\}\) – a solution to the subset problem.
5. Constraint Propagation Techniques (Semenov, Numerica, Jaulin, etc): Reminder

- Constraint propagation – traditional technique for solving constraint satisfaction problems.

- We start with the intervals \([x_1, \bar{x}_1], \ldots, [x_n, \bar{x}_n]\) containing the actual values of the unknowns \(x_1, \ldots, x_n\).

- On each iteration:
  - select \(i\) and a constraint \(f_j(x_1, \ldots, x_n) = 0\),
  - replace \([x_i, \bar{x}_i]\) with new interval \(x_i^{(j)} = [\bar{x}_i^{(j)}, \bar{x}_i^{(j)}] \overset{\text{def}}{=} \{x_i : x_i \in [\bar{x}_i, \bar{x}_i] \land f_j(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = 0\}
    for some \(x_k \in [\bar{x}_k, \bar{x}_k]\).

- If the process stalls, we bisect the interval for one the variables into two and try to decrease both resulting half-boxes.

- Problem: cannot use it if not all constraints are valid.
6. Traditional Interval-Related Constraint Propagation Techniques: Example

- **Toy problem:** find \( x \in [-5, 5] \) for which \( x - x^2 = 0 \).
- **Pre-processing:** parse the expression:
  \[
  r = x^2; x - r = 0.
  \]
- **Originally:** \( X = [-5, 5], R = [-\infty, \infty] \).
- **Use the first constraint:** \( x \in [-5, 5] \) implies \( r \in [0, 25] \), so for \( r \), the new interval is \( [-\infty, \infty] \cap [0, 25] = [0, 25] \):
  \[
  X = [-5, 5], \quad R = [0, 25].
  \]
- **Use the second constraint:** for \( x \), we have \( [-5, 5] \cap [0, 25] = [0, 5] \), and similarly for \( r \), so
  \[
  X = [0, 5], \quad R = [0, 5].
  \]
- **Use the first constraint:** \( x = \sqrt{r} \), hence
  \[
  X = [0, 2.24], \quad R = [0, 5].
  \]
- **Use the second constraint:**
  \[
  X = [0, 2.24], \quad R = [0, 2.24].
  \]
- After a while, we stall at \( X = R \approx [0, 1] \), so we bisect \( X \) to \( [0, 1/2] \) and \( [1/2, 1] \).
- Then, we converge to \( x = 0 \) and \( x = 1 \).
7. New Idea

- On each iteration, we still select a variable $x_i$, but:
  - instead of selecting a single constraint,
  - we try all $N$ constraints, and get $N$ resulting intervals $[x_i^{(j)}, \bar{x}_i^{(j)}]$.

- We know that $\geq N \cdot (1 - \varepsilon)$ constraints are satisfied.

- Hence $x_i \leq \bar{x}_i^{(j)}$ for $\geq N \cdot (1 - \varepsilon)$ different values $j$.

- Let us sort all $N$ upper endpoints $\bar{x}_i^{(j)}$ ($1 \leq j \leq N$) into an increasing sequence $u_1 \leq u_2 \leq \ldots \leq u_N$,

- Then we can guarantee that $x_i$ is smaller than (or equal to) at least $N \cdot (1 - \varepsilon)$ terms in this sequence.

- So, $x_i \leq u_N \cdot \varepsilon$.

- Similarly, if we sort the lower endpoints $\underline{x}_i^{(j)}$ into a decreasing sequence $l_1 \geq \ldots \geq l_N$, then $x_i \geq l_N \cdot \varepsilon$. 
8. **New Algorithm**

- On each iteration, we:
  - we select a variable $x_i$;
  - for each of $N$ constraints, we compute the corresponding interval $[\bar{x}_i^{(j)}, \bar{x}_i^{(j)}]$;
  - we sort all $N$ upper endpoints $\bar{x}_i^{(j)}$ ($1 \leq j \leq N$) into an increasing sequence $u_1 \leq u_2 \leq \ldots \leq u_N$,
  - we sort all $N$ lower endpoints $\underline{x}_i^{(j)}$ ($1 \leq j \leq N$) into a decreasing sequence $l_1 \geq l_2 \geq \ldots \geq l_N$, and
  - we take $[l_N \cdot \varepsilon, u_N \cdot \varepsilon]$ as the new interval for $x_i$.

- If the process stalls, we bisect the interval and try to decrease both resulting half-boxes.

- *Comment:* producing $u_{N \cdot \varepsilon}$ can be done faster than by sorting.
9. Other Potential Applications of the New Algorithm: Design and Control Problems

- In many areas of science and engineering, we are interested in solving *design* and *control* problems.

- *In mathematical terms*: a design or a control can be usually represented by the values of the relevant numerical parameters \( x = (x_1, \ldots, x_n) \).

- Usually, in these problems, the users describe several *constraints* that the desired design or control must satisfy.

- *Objective*: find a design (corr., a control) that satisfies all these constraints.
10. **How to Describe Constraints?**

- *Example:* an airplane design can be described in terms of:
  - the geometric parameters of the plane,
  - the thickness of the plates that form the airplane’s skin,
  - the weight and power of the engine, etc.

- *Typical constraint:* a limitation on some characteristics \( y = f(x_1, \ldots, x_n) \) of this design.

- *Examples*
  - the airplane’s speed must exceed some \( y_0 \),
  - its fuel use must not exceed a certain amount,
  - the overall cost must be within given limits.

- So, constraints are of the type \( f(x_1, \ldots, x_n) \leq y_0 \) or \( f(x_1, \ldots, x_n) \geq y_0 \) (or \( f(x_1, \ldots, x_n) = y_0 \)).
11. Constraint Satisfaction vs. Constrained Optimization

- *Constraint satisfaction*: find a design that satisfies given constraints.
- *Problem*:
  - different designs that satisfy the given constraints;
  - we must select one of these designs.
- Users can often describe their preference in terms of an *objective function* $g(x_1, \ldots, x_n)$ (whose value should be made as large as possible).
- *Constrained optimization*: maximizing $g(x_1, \ldots, x_n)$ under the given constraints.
- *In general*: both problem are NP-hard.
- *In practice*: there are many efficient tools for solving them.
12. “Soft” Constraints

- **Problem:** sometimes, the users constraints are inconsistent.

- **Example:** design a plane that is:
  - as fast and as fuel-efficient as the existing Airbus or Boeing planes,
  - but with 0 noise level.

- **Reasons for inconsistency:**
  - some constraints are *absolute* (e.g., safety constraints),
  - others are *desires* – they can be dismissed if not possible.

- Such “not required” constraints are called *soft constraints*.

- **Comment:** soft constraints are an important research topic, with annual conferences.

- **Idea:** when we cannot satisfy all the constraints, we should satisfy as many constraints as possible.
13. Case Study: Seismic Inverse Problem

- **Problem**: to determine the geophysical structure of a region.

- **Solution**: we:
  - measure seismic travel times, and
  - reconstruct velocities at different depths from this data.

- **Difficulty**: the inverse problem is ill-defined:
  - *large* changes in the original distribution of velocities can lead to
  - very *small* changes in the resulting measured values.

- **Conclusion**: many different velocity distributions are consistent with the same measurement results.
14. Drawbacks of the Existing Approach

- **Situation:** because of the non-uniqueness, the velocity distribution that is returned by the existing algorithm is usually not geophysically meaningful.

- **Example:** it predicts velocities outside of the range of reasonable velocities at this depth.

- **Current solution:** a geophysicist adjusts the initial approximation so as to avoid this discrepancy.

- **Problem:** several iterations are needed; it is very time-consuming.

- **Problem:** adjustment requires special difficult-to-learn skills.

- **Result:** the existing tools for solving the seismic inverse problem are not as widely used as they could be.
15. It Is Necessary to Take Expert Knowledge Into Consideration

- **Objective:** make the tools for processing seismic data more accessible.
- **Solution:** incorporate the expert knowledge into the algorithm for solving the inverse problem.
- **Example why expert knowledge is needed:** velocity is outside the interval of values which are possible at this depth for this particular geological region.
- **Corresponding expert knowledge:** the intervals of possible values of data.
- **What needs to be done:** modify the inverse algorithms in such a way that the velocities are always within these intervals.
- **Question:** how can we do it?
16. How We Can Use Interval Uncertainty

- How algorithms work now:
  - start with a reasonable velocity model;
  - predict travel times \( x_i \) between stations;
  - use the difference \( \Delta x_i = \tilde{x}_i - x_i \), where \( \tilde{x}_i \) are measured values, to adjust the velocity model:
    * divide \( \Delta x_i \) by the length \( L \) of the path;
    * add \( \Delta x_i / L \) to all slownesses along the path.

- How to modify when we know the interval \([s_j, \bar{s}_j]\) of possible slownesses:
  - first, we compute the next approximation \( s_j^{(k)} \) to the slownesses,
  - then, we replace \( s_j^{(k)} \) with the nearest value within the interval \([s_j, \bar{s}_j]\).
17. Explicit Expert Knowledge: Fuzzy Uncertainty

- Experts can usually produce a wider interval of which they are practically 100% certain.

- In addition, experts can also produce narrower intervals about which their degree of certainty is smaller.

- As a result, instead of a single interval, we have a nested family of intervals corresponding to different levels of uncertainty.

- In effect, we get a fuzzy interval (of which different intervals are $\alpha$-cuts).

- Previously: a solution is satisfying or not.

- New idea: a satisfaction degree $d$.

- Specifics: $d$ is the largest $\alpha$ for which all $s_i$ are within the corresponding $\alpha$-cut intervals.
18. Implicit Expert Knowledge: Interval Uncertainty

- **Situation:** sometimes, velocities are in the interval, but the geophysical structure is still not right.

- **Explanation:**
  - algorithms assume that the measured errors are independent and normally distributed;
  - so, stopping criterion is MSE \( E \overset{\text{def}}{=} \sum_{i=1}^{N} (x_i - \tilde{x}_i)^2 \);
  - for geophysically meaningless models, \( E \) is small, but some differences \( x_i - \tilde{x}_i \) are large.

- **Solution:** require that \( |x_i - \tilde{x}_i| \leq \Delta \) for some bound \( \Delta \).
19. How We Can Use Interval Uncertainty

- **Problem**: how can we guarantee that we only get solutions which are physical in the above sense?

- **Traditional approach**: once the mean square error is small, we stop iterations.

- **Natural new idea**: continue iterations until all (or rather almost all, with proportion $\geq 1 - \varepsilon$) differences $|x_i - \tilde{x}_i|$ are under $\Delta$.

- **Question**: what if this does not happen?

- **Similar question**: what traditional algorithms do if we do not MSE small?

- **Answer to similar question**: restart computations with a different starting velocity model.

- **Solution to our problem**: restart computations with a different starting velocity model.
20. A General Problem

- Inverse problem is ill-posed $\approx$ has many different solutions.
- Many inverse problems in science and engineering are ill-posed.
- *Regularization:* we select a solution with a certain property, e.g., a smooth one, $J \overset{\text{def}}{=} \int (x'(t))^2 \, dt \to \min$.

- *Discrete case:* $J_{\text{discr}} \overset{\text{def}}{=} \sum_i (x(t_{i+1}) - x(t_i))^2$.

- *2-D case:* $J \overset{\text{def}}{=} \sum_{n_1,n_2} [(f(n_1+1,n_2) - f(n_1,n_2))^2 + (f(n_1,n_2+1) - f(n_1,n_2))^2]$, or, equivalently, $J = \sum_{p,p'} (f(p) - f(p'))^2$.

- Smoothness leads to efficient algorithms.

- *Problem:* for inverse problem in geophysics, we only have piecewise smoothness.
21. General Problem: Precise Formulation

- **Idea:** we only take into account the pairs of neighboring pixels that belong to the same zone:
  \[ J(Z) = \sum_{p,p' \text{ are neighbors in the same zone}} (f(p) - f(p'))^2, \]
  where \( Z \) denotes the information about the zones.

- Often, we do not know where the edges are, i.e., we do not know \( Z \).

- **Idea:** find \( Z \) for which the result inside each zone is the smoothest, i.e., minimize
  \[ J^* = \min_{\text{all possible divisions } Z \text{ into zones}} J(Z). \]

- **Problem:** the resulting problem is no longer convex.

- It is known that non-convex problems are, in general, more computationally difficult.
22. **Result: Reconstructing Piecewise Smooth Solutions is NP-Hard**

- *Idea of the proof:* we reduce a known NP-hard problem (subset sum) to our problem.

- *Subset sum:*
  - given $m$ positive integers $s_1, \ldots, s_m$ and an integer $s > 0$,
  - check whether it is possible to find a subset of this set of integers whose sum is equal to exactly $s$.

- *Alternative description:* check whether there exist $x_i \in \{0, 1\}$ for which $\sum s_i \cdot x_i = s$. 
23. Reduction

- We want to reconstruct an $m \times m$ solution $f(n_1, n_2)$.
- Let $d = \lfloor m/2 \rfloor$. We want a piecewise smooth solution $f(n_1, n_2)$ that consists of two zones.
- The following linear constraints describe the consistency between the observations and the desired solution:
  - $f(n_1, n_2) = 1$ for $n_2 > d$;
  - $\sum_{i=1}^{m} s_i \cdot f(i, d) = s$; and
  - $f(n_1, n_2) = 0$ for $n_2 < d$.
- **Problem**: among all the solutions that satisfy these constraints, find the one with the smallest non-smoothness $J^*$. 
24. Proof

- Let us show that \( \min J^* = 0 \) \( \leftrightarrow \) the original subset problem has a solution.
- If \( J^* = 0 \), then all the values within each zone must be the same.
- Since \( f = 1 \) for \( n_2 > d \) and \( f = 0 \) for \( n_2 < d \), every value \( f(n_1, n_2) \) is \( 1 \) or \( 0 \).
- Thus, the values \( x_i = f(i, d) \in \{0, 1\} \) solve the original subset problem
\[ \sum s_i \cdot x_i = s. \]
- Vice versa:
  - if the selected instance of the original subset problem has a solution \( x_i \),
  - then we can take \( f(i, d) = x_i \) and get the solution of the inverse problem
    for which the degree of non-smoothness is exactly 0.
25. Acknowledgments

This work was supported in part:

- by NASA under cooperative agreement NCC5-209,
- by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, and
- by NIH grant 3T34GM008048-20S1.
26. In Many Practical Problems, It Is Very Important to Test Smoothness

- In many practical problems, we must check whether a given object is smooth or whether it has non-smooth areas:
  - *aerospace structure*: cracks, holes, other faults;
  - *mammography*: small clots, cracks, etc., which may indicate a tumor.

- **Smoothness leads to linearity**
  - If a tested structure has no faults, then the surface is usually smooth.
  - As a result, the dependencies $f_i$ between the test signals $x_j$ and the received signals $y_i$ are also smooth.
  - Since we are sending relatively weak signals $x_i$ (strong signals can damage the plane), we can neglect quadratic (and higher order) terms in Taylor series and only consider linear terms in these series.

- **Non-smoothness leads to non-linearity**
  - A fault (e.g., a crack) is, usually, a violation of smoothness.
  - Thus, if there is a fault, the structure stops being smooth; hence, the function $f_j$ stops being smooth.
  - Therefore, linear terms are no longer sufficient.

- So, we can detect the fault by checking whether the dependency between $y_j$ and $x_i$ is linear.
27. The Resulting Proposal: Main Idea

As a result of the above analysis, we propose the following way of detecting faults:

- We apply different signals $x_j$ to the object, and measure the response $y_i$.
- If the measurement results are consistent with the linear dependence of $y_i$ on $x_j$, this means that there are no faults, and no further testing is needed.
- If the measurement results are inconsistent with the linear model, this means that there is a fault, and so further thorough tests are needed.

This proposal saves time and resources:

- Checking linearity is easy.
- As a result, for non-destructive evaluation of aerospace structures, we get a simple test that:
  - enables us to save time and resources (necessary for the detailed solution of the inverse problem)
  - by limiting this detalization only to the cases when the presence of the faults was revealed by non-linearity.
28. Mechanical Analysis of the Problem

• Fault-less plate:
  – Transmitter sends a signal \( x(t) = A \cdot \cos(\omega \cdot t) \).
  – This signal travels to the receiver (at a distance \( d \)) with a speed of sound \( v \), and thus gets delayed by \( \Delta t = d/v \).
  – Hence, the received signal is
    \[
    y(t) = k \cdot x(t - \Delta t) = k \cdot A \cdot \cos(\omega \cdot t - \omega \cdot d/v),
    \]
    where the coefficient \( k \) describes the loss of amplitude.
  Thus, for a fault-less plate, we indeed have a linear dependence between the transmitted signal \( x(t) \) and the measured signal \( y(t) \).

• Plate with faults:
  – For a plate with a crack, \( \Delta t = d/v + d_0/v_0 \), where \( d_0 \) is the linear size of the fault, and \( v_0 \) is the speed of sound in the air.
  – As we transmit the signal \( x(t) \), the plate starts vibrating.
  – This vibration changes the position of both borders of the crack and therefore, changes (harmonically) the distance \( d_0: d_0 = d_0(t) \).
  – So, we get a non-linear phase:
    \[
    y(t) = k \cdot A \cdot \cos(\omega \cdot t - \omega \cdot d/v - \omega \cdot d_0(t)/v_0),
    \]
    and hence a non-linear dependence.
29. Experimental Confirmation of Non-Linearity: Pseudo-Random Signals

For *pseudo-random signals* $x(t)$ (which combine components of several different frequencies with pseudo-random amplitudes and pseudo-random phases):

- For a fault-less plate, the dependence between the transmitted signal $x(t)$ and the measured signal $y(t)$ is linear, i.e.,
  \[ y(t) = \int A(t - s) \cdot x(s) \, ds \]
  for some function $A(t)$.

- For a plate with a fault, this dependence is non-linear: namely, cubic terms must be taken into consideration.

The amplitude of the cubic term is roughly proportional to the cube of the linear fault size. Thus:

- not only the non-linear terms indicate the *presence* of the fault, but also
- the value of the cubic term can be used to determine the *size* of the fault.
30. Pseudo-Random Signals Are Difficult to Generate

- In practice, it is difficult to generate pseudo-random signals.
- It is therefore desirable to confirm that non-linearity can be also observed for simpler signals, e.g., for sinusoid signals.
- In our experiment, as a signal \( x_j \), we sent an ultrasound wave. To generate this wave:
  - a sinusoid electric signal \( x(t) = A \cdot \cos(\omega \cdot t) \) was sent to the transducer,
  - which then generated an ultrasonic wave in the tested object.
- The transducer was set at an angle of incidence of 31° with the plate, so that a wave would go along the surface of the plate (such waves are called Lamb waves).
- The transducer is somewhat non-linear.
- To separate the non-linearity of the transducer from the non-linearity of the plate itself, we placed two sensors on the plate:
  - the first sensor is located near the transducer, and it measures the ultrasonic wave \( x_1(t) \) that the transducer generates;
  - the second sensor is located at a distance from the transducer, and it measure the wave \( x_2(t) \) changed after passing through the plate.
- Then, we check whether \( x_2(t) \) linearly depends on \( x_1(t) \).
31. Experiment with Sinusoid Signals: A Software Part

- After measuring the two signals \( x_1(t) \) and \( x_2(t) \), we apply FFT to both.
- Compute the total energy \( E_1 = \int |\hat{x}_1(\omega)|^2 d\omega \) of the signal \( x_1 \) in the frequency range [350 KHz, 650Khz] of the transducer.
- Compute the total energy \( E_2 = \int |\hat{x}_2(\omega)|^2 d\omega \) of the signal \( x_2 \) in the same frequency range.
- Check whether \( E_2 \) is a linear function of \( E_1 \), i.e., whether there exist \( k \) and \( n \) for which
  \[
  E_2 = k \cdot E_1 + n.
  \]
- Due to inevitable measurement inaccuracy, after each measurement, we do not get the exact values \( E_i(V) \).
- We only get an interval \([E_i^-(V), E_i^+(V)]\) of possible values of \( E_i(V) \).
- The question is: is this data consistent with the assumption that \( E_2(V) \) is a linear function of \( E_1(V) \)?
- In other words, it is possible to find real numbers \( k > 0 \), \( n \), and values \( E_1(V) \in [E_1^-(V), E_1^+(V)] \) and \( E_2(V) \in [E_2^-(V), E_2^+(V)] \) for which
  \[
  E_2(V) = k \cdot E_1(V) + n?
  \]
32. **Taking Measurement Inaccuracy into Consideration: Solution**

- **Proposition:** The set of intervals \([E_1^-(V), E_1^+(V)], [E_2^-(V), E_2^+(V)]\) is consistent with the assumption that \(E_2(V)\) is a linear function of \(E_1(V)\) if and only if the following inequality is true:

\[
\max_{V'<V} \frac{E_2^-(V) - E_2^+(V')}{E_1^+(V) - E_1^-(V')} \leq \max_{V'<V} \frac{E_2^+(V') - E_2^-(V)}{E_1^+(V') - E_1^-(V)}.
\]

- **Algorithm:** To check non-linearity, we must check the above inequality.

- **Practical recommendation:** Brief summary: To detect the faults, we must use at least two different signal levels.

  - If the increase in the signal level \(x_j\) leads to a proportional increase in the measured values \(y_i\), then most probably the object is smooth.

  - If the dependence of \(y_i\) on \(x_j\) is non-linear, then, most probably, there is a fault, so further analysis is needed.
33. Experimental Results

- **Undamaged case:***

<table>
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<th>V</th>
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<th>$[E_2^-(V), E_2^+(V)]$</th>
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<tr>
<td>9V</td>
<td>[4.59, 4.69]</td>
<td>[2.70, 2.80]</td>
</tr>
</tbody>
</table>

  In the undamaged case, we clearly have a linear dependency:

  $$E_2(V) \approx 0.6 \cdot E_1(V).$$

- **Damaged case:***

<table>
<thead>
<tr>
<th>V</th>
<th>$[E_1^-(V), E_1^+(V)]$</th>
<th>$[E_2^-(V), E_2^+(V)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>damaged, $10^5$</td>
<td>damaged, $10^4$</td>
</tr>
<tr>
<td>0V</td>
<td>[0.02, 0.03]</td>
<td>[0.06, 0.11]</td>
</tr>
<tr>
<td>6V</td>
<td>[0.69, 0.70]</td>
<td>[0.23, 0.28]</td>
</tr>
<tr>
<td>7V</td>
<td>[0.87, 0.92]</td>
<td>[0.14, 0.23]</td>
</tr>
<tr>
<td>8V</td>
<td>[1.05, 1.08]</td>
<td>[4.75, 4.84]</td>
</tr>
<tr>
<td>9V</td>
<td>[1.28, 1.32]</td>
<td>[5.57, 5.80]</td>
</tr>
</tbody>
</table>

  In the damaged case, the dependence is clearly non-linear.