Intelligence Techniques Are Needed to Further Enhance the Advantage of Groups with Diversity in Problem Solving

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1. Introduction to the Problem

- **Empirical fact:** diversity in a group often enhances the group’s ability to solve problems.

- **Theoretical explanation** (S. E. Page): diverse groups *can* outperform groups of high-ability problem solvers.

- **Problem:** algorithmic diversity rules (like quotas) are not always successful.

- **Our approach:** we consider the problem of designing the most efficient group as an optimization problem.

- **Our result:** this optimization problem is computationally difficult (NP-hard).

- **Conclusion:** it is not possible to come up with simple algorithmic rules for designing such groups.

- **Conclusion:** we must combine standard optimization techniques with expert knowledge.
2. Towards the Formulation of the Problem in Exact Terms

- From $n$ individuals $\{1, \ldots, n\}$, we must select the most efficient group $G \subseteq \{1, \ldots, n\}$ for solving the problem.
- For each $i$, we set $x_i = 1$ is the $i$-th person is selected, and $x_i = 0$ otherwise.
- For simple mechanical work, group efficiency is the sum of productivities: $p = \sum_{i \in G} p_i = \sum_{i=1}^{n} p_i \cdot x_i$.
- For more complex tasks, interaction can either help ($p_{ij} > 0$) or inhibit efficiency ($p_{ij} < 0$).
- After the linear approximation, the next approximation is quadratic:

$$p = \sum_{i=1}^{n} p_i \cdot x_i + \sum_{i \neq j} p_{ij} \cdot x_i \cdot x_j.$$
3. Explanations: Why Help and Why Inhibition

- **Homogeneous group**: individuals with similar ways of thinking and with similar skills.
- **Property**: there is not much that these individuals can learn from each other.
- **Simple case**: the problem is easy to subdivide into sub-problems.
- **Typical case**: the problem is not easy to subdivide.
- **Result**: the solvers follow similar paths, duplicate work.
- **Productivity**: same as for one solver: \( p \approx p_i < p_i + p_j \), i.e., \( p_{ij} < 0 \).
- **Diverse group**: individuals complement each other, learn from each other.
- **Result**: productivity increases: \( p > p_i + p_j \), i.e., \( p_{ij} > 0 \).
4. Problem of Selecting the Most Efficient Group: Precise Formulation

- **Given:**
  - an integer \( n > 0 \);
  - rational numbers \( p_1, \ldots, p_n \), and
  - rational numbers \( r_{ij}, 1 \leq i, j \leq n, i \neq j \).

- **Find:** the combination of \( n \) values \( x_1 \in \{0, 1\}, \ldots, x_n \in \{0, 1\} \) for which the expression

\[
p = \sum_{i=1}^{n} p_i \cdot x_i + \sum_{i \neq j} p_{ij} \cdot x_i \cdot x_j
\]

is the largest possible.

- **Our result:** the problem of selecting the most efficient group is NP-hard.
5. Maybe a Less Ambitious Problem Will Be Easier to Solve?

- **We wanted:** the most efficient group, with the largest possible productivity.

- **Problem:** the related task is computationally difficult (NP-hard).

- **Natural idea:** maybe a less ambitious problem will be easier to solve.

- **Specific suggestion:**
  - instead of looking for a group with the largest possible productivity,
  - look for a group with a desired level of productivity $p \geq p_0$.

- **Problem:** as we will see, this “relaxed” problem is still computationally difficult (NP-hard).
6. Problem of Selecting a Group with a Given Efficiency

• **Given:**
  
  – an integer \( n > 0; \)
  – rational numbers \( p_1, \ldots, p_n; \)
  – rational numbers \( r_{ij}, 1 \leq i, j \leq n, i \neq j, \) and
  – a rational value \( p_0. \)

• **Find:** the combination of \( n \) values \( x_1 \in \{0, 1\}, \ldots, x_n \in \{0, 1\} \) for which

\[
p \overset{\text{def}}{=} \sum_{i=1}^{n} p_i \cdot x_i + \sum_{i \neq j} p_{ij} \cdot x_i \cdot x_j \geq p_0.
\]

• **Our result:** The problem of selecting a group with a given efficiency is NP-hard.
7. A Simple Illustrative Example

- **Description:** we have two groups of equally productive people, $G_1 = \{1, \ldots, m\}$ and $G_2 = \{m + 1, \ldots, 2m\}$:
  
  $$p_1 = \ldots = p_m = p_{m+1} = \ldots = p_{2m} = 1.$$

- **Help:** persons from the same group collaborate: $p_{ij} = 1$ for $i, j \leq m$ or $i, j > m$.

- **Inhibition:** between-group tension decreases productivity: $p_{ij} = p_{ji} = -a$ for $i \leq m$ and $j > m$.

- **We select:** $m_1$ folks from $G_1$ and $m_2$ folks from $G_2$.

- **Resulting productivity:** $p = m_1^2 + m_2^2 - a \cdot m_1 \cdot m_2$, where $m_i \in [0, m]$.

- **Solution** (Case $a < 1$): the most diverse group $m_1 = m_2 = m$ is the most efficient.

- **Comment:** when $a > 1$, tensions are so high that homogeneous groups are better: e.g., $m_1 = m$ and $m_2 = 0$. 
8. Conclusions

• One of the objectives of fuzzy techniques: to formalize the meaning of words from natural language.

• Examples: “efficient”, “diverse”, etc.

• The main use of fuzzy techniques: formalize expert knowledge expressed in natural language.

• In this paper, we have shown that
  – if we do not use this knowledge, i.e., if we only use the data,
  – then selecting the most efficient group is computationally difficult (NP-hard).

• Thus, the need to select efficient groups in reasonable time justifies the use of fuzzy (intelligent) techniques.

• Moreover, there is a need to combine intelligent techniques with more traditional optimization techniques.
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10. When is an Algorithm Feasible?

- The notion of NP-hardness is related to the known fact that some algorithms are feasible and some are not.
- Whether an algorithm is feasible or not depends on how many computational steps it needs.
- **Case 1:** for some input \( x \) of length \( \text{len}(x) = n \), an algorithm requires \( 2^n \) computational steps.
- **Example:** for an input of a reasonable length \( n \approx 300 \), we need \( 2^{300} \) computational steps.
- **Problem:** this takes longer than the Universe’s lifetime.
- **Conclusion:** this algorithm is not feasible.
- **Case 2:** an algorithm requiring \( n^2 \) or \( n^3 \) steps is usually feasible.
- **Resulting definition:** an algorithm is feasible if its running time \( t(n) \) is bounded by a polynomial \( P(n) \).
11. NP: Class of General Problems

- **General formulation:**
  - we have some information $x$;
  - we need to find $y$ which satisfies the feasible-to-check property $R(x, y)$.

- **Example from mathematics:**
  - given: a mathematical statement $x$;
  - find: a proof $y$ of $x$ or of “not $x$”.

- **Comment:** computers can easily check step-by-step proofs $y$, but finding a proof is a challenge.

- **Engineering:** find a design $y$ that satisfies given specifications $x$.

- **Physics:** find a formula $y$ that is consistent with all the observations $x$. 
12. Class NP and the Notion of NP-Hardness

- Every problem from the class NP can be solved by exhaustively checking all $2^n$ possible solutions $y$.
- Example: try all possible combinations of $n$ symbols until we find a proof.
- Open question: it is not known whether a feasible algorithm can solve all NP problems.
- What is known: some NP problems are more difficult than others ("NP-hard").
- Precise meaning: every problem from NP can be reduced to this problem.
- How to prove NP-hardness: reduce one of the known NP-hard problems $P_k$ to the desired one $P_d$.
- Proof: every $P \in$ NP can be reduced to $P_k$, and $P_k$ can be reduced to $P_d$, so $P$ can be reduced to $P_d$. 

13. Proof of Our Result: Main Ideas

- We prove NP-hardness of our problem by reducing the following known NP-hard problem to it.

- The *subset sum* problem:
  
  - *given:* $n$ positive integers $s_1, \ldots, s_n$;
  
  - *find:* the signs $\varepsilon_i \in \{-1, 1\}$ for which $\sum_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.

- *Reduction* means that:
  
  - to every instance $s_1, \ldots, s_n$ of the subset sum problem,
  
  - we must assign (in a feasible, i.e., polynomial-time way) an instance $p_0, p_i, p_{ij}$ of our problem,
  
  - in such a way that the solution to the new instance will lead to the solution of the original instance.
14. Proof (cont-d)

- **Original problem:** find $\varepsilon_i \in \{-1, 1\}$ s.t. $\sum_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.

- **New problem:** find $x_i \in \{0, 1\}$ for which

  $$p \overset{\text{def}}{=} \sum_{i=1}^{n} p_i \cdot x_i + \sum_{i \neq j} p_{ij} \cdot x_i \cdot x_j \geq p_0.$$ 

- A solution $x_i \in \{0, 1\}$ of the new instance must lead to the solution $\varepsilon_i \in \{-1, 1\}$ of the original instance.

- **Natural idea:** take $\varepsilon_i = 2 \cdot x_i - 1$.

- **Natural reduction:** take $p = p_0 - \left(\sum_{i=1}^{n} \varepsilon_i \cdot s_i\right)^2$.

- **Why it works:** $p \geq p_0 \iff \sum_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.

- **Specifics:** $p_0 = \sum_{i=1}^{n} s_i$, $p_i = 4 \cdot s_0 \cdot s_i - 4 \cdot s_i^2$, $p_{ij} = -4 \cdot s_i \cdot s_j$. 
15. Discussion

- Strictly speaking, we have proved NP-hardness of a specific choice of the quadratic function $p(x_1, \ldots, x_n)$.
- However, one can easily check that
  - if a problem $P_0$ is NP-hard,
  - then a more general problem $P_1$ is NP-hard as well.
- Thus, we have indeed proved that the (more general) problem is also NP-hard.