r-Bounded Fuzzy Measures are Equivalent to $\varepsilon$-Possibility Measures

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1. Outline

- Traditional probabilistic description of uncertainty is based on *additive* probability measures.

- To describe non-probabilistic uncertainty, it is therefore reasonable to consider *non-additive* measures.

- An important class of non-additive measures are *possibility* measures, for which \( \mu(A \cup B) = \max(\mu(A), \mu(B)) \).

- In this talk, we show that possibility measures are, in some sense, universal approximators:
  - for every \( \varepsilon > 0 \),
  - every non-additive measure which satisfies a certain reasonable boundedness property
  - is equivalent to a measure which is \( \varepsilon \)-close to a possibility measure.
2. Additive Measures: Brief Reminder

- Let $X$ be a set called a *universal set*.

- An *algebra* $\mathcal{A}$ is a non-empty class of subsets $A \subseteq X$ which is closed under complement, $\cup$, and $\cap$, i.e.:
  - if $A \in \mathcal{A}$, then its complement $-A$ is also in $\mathcal{A}$;
  - if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$;
  - if $A \in \mathcal{A}$ and $B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

- An *additive measure*, we mean a function $\mu$ mapping sets $A \in \mathcal{A}$ to numbers $\mu(A) \geq 0$ s.t. $A \cap B = \emptyset$ implies
  \[ \mu(A \cup B) = \mu(A) + \mu(B). \]

- *Examples*: length, area, volume, probability.

- *Properties*: monotonicity ($A \subseteq B$ implies $\mu(A) \leq \mu(B)$), $\mu(\emptyset) = 0$. 
3. Fuzzy Measures: Brief Reminder

• A function $\mu(A)$ defined on an algebra of subsets of a universal set $X$ is called a fuzzy measure if:
  
  – it is monotonic, i.e., $A \subseteq B$ implies $\mu(A) \leq \mu(B)$, and
  
  – it satisfies the properties $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

• A function $\mu(A)$ is called maxitive if for every two sets $A$ and $B$, we have $\mu(A \cup B) = \max(\mu(A), \mu(B))$.

• By a possibility measure, we mean a maxitive fuzzy measure.
4. Which Measures Are Reasonable?

- In general, a measure $\mu(A)$ describes how important is the set $A$.
- The larger the measure, the more important is the set $A$.
- From this viewpoint:
  - if we take the union $A \cup B$ of two sets of bounded size,
  - then the size of the union cannot be arbitrarily large.
- The size of $A \cup B$ should be limited by some bound depending on the bound on $\mu(A)$ and $\mu(B)$.
- Similarly, if the sizes of $A$ and $B$ are sufficiently small, then the size of the union should also be small.
5. Which Measures Are Reasonable: Definition

- By a non-additive measure, we mean a function $\mu(A)$ that assigns, to each set $A \in \mathcal{A}$, a number $\mu(A) \geq 0$.

- A non-additive measure $\mu$ is $r$-bounded if it satisfies the following two properties:
  - for every $\Gamma > 0$, there exists a $\Delta > 0$ such that if $\mu(A) \leq \Gamma$ and $\mu(B) \leq \Gamma$, then $\mu(A \cup B) \leq \Delta$;
  - for every $\eta > 0$, there exists a $\nu > 0$ such that if $\mu(A) \leq \nu$ and $\mu(B) \leq \nu$ then $\mu(A \cup B) \leq \eta$.

- Comment: every additive measure is $r$-bounded, with $\Delta = 2 \cdot \Gamma$ and $\nu = \frac{\eta}{2}$. 
6. **Equivalence: Analysis of the Problem**

- The numerical values of probabilities have observable sense.

- The probability of an event $E$ can be defined as the limit of the frequency with which $E$ occurs.

- In contrast, e.g., possibility values do not have direct meaning.

- The only important thing is which values are larger and which are smaller.

- This describes which events are more possible and which are less possible.

- From this viewpoint,
  - if two measures can be obtained from each other by a transformation that preserves the order,
  - such measures can be considered to be equivalent.
7. Definitions and the Main Result

- Non-additive measures $\mu(A)$ and $\mu'(A)$ are called equivalent if there exists a 1-1 monotonic function $f(x)$ s.t.
  \[ \mu'(A) = f(\mu(A)) \]
  for every $A$.

- Let $\varepsilon > 0$ be a real number.

- A non-additive measure $\mu(A)$ is an $\varepsilon$-possibility measure if for every $A$ and $B$:
  \[ \max(\mu(A), \mu(B)) \leq \mu(A \cup B) \leq (1 + \varepsilon) \cdot \max(\mu(A), \mu(B)). \]

- **Result**: For every $\varepsilon > 0$, every $r$-bounded non-additive measure is equivalent to an $\varepsilon$-possibility measure.

- Can we strengthen this result? Is each $r$-bounded measure equivalent to a possibility measure?

- A simple answer is “No”: any measure which is equivalent to a maxitive one is also maxitive.
8. First Auxiliary Result: Possibility of Uniform Equivalence

- We proved that each r-bounded non-additive measure can be re-scaled into an “almost” possibility measure.
- Sometimes, we have several measures.
- Can we re-scale all of them into $\varepsilon$-possibility measures by using the same re-scaling $f(x)$?
- \textit{Result:} For every $\varepsilon > 0$, and for every finite set of r-bounded non-additive measures $\mu_1(A), \ldots, \mu_n(A)$,
  - there exists a 1-1 function $f(x)$ for which
  - all $n$ measures $\mu'_i(A) \overset{\text{def}}{=} f(\mu_i(A))$ are $\varepsilon$-possibility measure.
9. Case of Generalized Metric

• Similarly to the fact that measures describe size, metrics describe distance.

• Usually, we consider metrics $d(a, b)$ which satisfy the triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$.

• However, it does not have to be this particular inequality. What is important is that:
  – if $d(a, b)$ and $d(b, c)$ are bounded by some $\Gamma > 0$,
  – then the distance $d(a, c)$ cannot be arbitrarily large,
  – it should be limited by some bound depending on the bound on $d(a, b)$ and $d(b, c)$.

• Similarly:
  – if $d(a, b)$ and $d(b, c)$ are sufficiently small,
  – then the distance $d(a, c)$ is also small.
10. Generalized Metric (cont-d)

- A function \( d : X \times X \to R_0^+ \) is called an \textit{r-bounded metric} if it satisfies the following two properties:
  - for every \( \Gamma > 0 \), there exists a \( \Delta > 0 \) such that if \( d(a, b) \leq \Gamma \) and \( d(b, c) \leq \Gamma \), then \( d(a, c) \leq \Delta \);
  - for every \( \eta > 0 \), there exists a \( \nu > 0 \) such that if \( d(a, b) \leq \nu \) and \( d(b, c) \leq \nu \) then \( d(a, c) \leq \eta \).

- A function \( d : X \times X \to R_0^+ \) is called an \textit{ultrametric} if
  \[
d(a, c) \leq \max(d(a, b), d(b, c)) \text{ for all } a, b, \text{ and } c.
  \]

- \( d : X \times X \to R_0^+ \) is called an \textit{\( \varepsilon \)-ultrametric} if
  \[
d(a, c) \leq (1 + \varepsilon) \cdot \max(d(a, b), d(b, c)) \text{ for all } a, b, \text{ and } c.
  \]
11. Generalized Metrics: Results

- r-bounded metrics $d(a, b)$ and $d'(a, b)$ are called *equivalent* if there exists a 1-1 monotonic $f(x)$ s.t.:
  \[ d'(a, b) = f(d(a, b)) \text{ for all } a, b. \]

- **Result:** For every $\varepsilon > 0$, every r-bounded metric is equivalent to an $\varepsilon$-ultrametric.

- For every $\varepsilon > 0$, and for every finite set of r-bounded metrics $d_1(a, b), \ldots, d_n(a, b)$, there is a 1-1 f-n $f(x)$ s.t.
  \[ d'_i(a, b) \overset{\text{def}}{=} f(d_i(a, b)) \text{ are } \varepsilon\text{-ultrametrics for all } i. \]
12. General Result Including Measures and Metrics as Special Cases

- By a *domain*, we mean a set $S$ with a partial binary operation $\circ : S \times S \to S$.
- For measures, $S$ is an algebra of sets, and $\circ$ is the union.
- For metrics, $S$ is the set of all pairs $(a, b)$, and the binary operation transforms $(a, b)$ and $(b, c)$ into $(a, c)$.
- By a *characteristic*, we mean a function $F : S \to R_0^+$. A characteristic $F(x)$ is called *$r$-bounded* if:
  - $\forall \Gamma > 0 \exists \Delta > 0$ s.t. if $F(x) \leq \Gamma$, $F(x') \leq \Gamma$, and $x \circ x'$ is defined, then $F(x \circ x') \leq \Delta$;
  - for every $\eta > 0$, there exists a $\nu > 0$ such that if $F(x) \leq \nu$ and $F(x') \leq \nu$ then $F(x \circ x') \leq \eta$. 
13. General Results (cont-d)

- $F(x)$ is called an $\varepsilon$-maxitive if for all $x$ and $x'$ for which $x \circ x'$ is defined,

$$F(x \circ x') \leq (1 + \varepsilon) \cdot \max(F(x), F(x')).$$

- $F(x)$ and $F'(x)$ are called equivalent if there exists a 1-1 monotonic f-n $f(x)$ for which

$$F'(x) = f(F(x)) \text{ for all } x \in S.$$

- **Result:** for every $\varepsilon > 0$, every $r$-bounded characteristic is equivalent to an $\varepsilon$-maxitive one.

- **Result:** for every $\varepsilon > 0$, and for every finite set of $r$-bounded characteristics $F_1(x), \ldots, F_n(x)$,

  - there exists a 1-1 f-n $f(x)$ for which

  - all $n$ characteristics $F'_i(x) \overset{\text{def}}{=} f(F_i(x))$ are $\varepsilon$-maxitive.
14. Conclusions

- The traditional probabilistic description of uncertainty uses additive probability measures.
- For describing non-probabilistic uncertainty, it is therefore reasonable to use non-additive measures.
- The most well-known example of such measures are possibility measures $\mu(A)$, for which:
  \[
  \mu(A \cup B) = \max(\mu(A), \mu(B)) \quad \text{for all } A \text{ and } B.
  \]
- In this talk, we show that the wide use of possibility measures may be explained by the fact that:
  - under some reasonable conditions,
  - these measures can approximate any non-additive measures.
15. Conclusions (cont-d)

- Namely, for every $\varepsilon > 0$, each non-additive measure is isomorphic to an $\varepsilon$-possibility measure, s.t.:
  \[
  \max(\mu(A), \mu(B)) \leq \mu(A \cup B) \leq (1+\varepsilon) \cdot \max(\mu(A), \mu(B)).
  \]

- If we have several measures, then:
  - the tuple consisting of these measures
  - is isomorphic to a tuple of $\varepsilon$-possibility measures.

- Similar results are also proven for generalized metrics.
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17. Proof of the Main Result

- Let us first define a doubly infinite sequence \( \ldots < c_{-1} < c_0 < c_1 < \ldots \) as follows.
- We take \( c_0 = 1 \); once we have defined \( c_k \) for some \( k \geq 0 \), we define \( c_{k+1} \) as follows.
- By def-n of an \( r \)-bounded measure, there exists \( \Delta_k > 0 \) s.t. if \( \mu(A) \leq c_k \) and \( \mu(B) \leq c_k \), then \( \mu(A \cup B) \leq \Delta_k \).
- We then take \( c_{k+1} \overset{\text{def}}{=} (1 + \varepsilon) \cdot \max(c_k, \Delta_k) \).
- Here, \( c_0 = 1 \) and \( c_{k+1} \geq (1 + \varepsilon) \cdot c_k \); thus, \( c_k \geq (1 + \varepsilon)^k \) and \( c_k \to \infty \) when \( k \) increases.
- Similarly, once we have defined the value \( c_{-k} \) for some \( k \geq 0 \), we define \( c_{-(k+1)} \) as follows.
- By def-n of an \( r \)-bounded measure, there exists \( \nu_k > 0 \) s.t. if \( \mu(A) \leq \nu_k \) and \( \mu(B) \leq \nu_k \), then \( \mu(A \cup B) \leq c_{-k} \).
- We then take \( c_{-(k+1)} \overset{\text{def}}{=} (1 - \varepsilon) \cdot \min(c_{-k}, \nu_k) \).
18. Proof (cont-d)

• Here, $c_0 = 1$ and $0 < c_{-(k+1)} \leq (1 - \varepsilon) \cdot c_{-k}$, hence $0 < c_{-k} \leq (1 - \varepsilon)^k$ and $c_{-k} \to 0$ when $k \to \infty$.

• Since $c_k \uparrow$, $c_k \to \infty$, and $c_{-k} \to 0$, for every $x > 0$, $\exists k$ s.t. $c_{k-1} < x \leq c_k$; we define $f(x)$ as follows:
  
  – for each integer $k$, we take $f(c_k) = (1 + \varepsilon)^{k/2}$ and
  – for each value $x$ between $c_{k-1}$ and $c_k$, we define $f(x)$ by linear interpolation: if $c_{k-1} < x \leq c_k$, then

  $$f(x) = f(c_{k-1}) + \frac{x - c_{k-1}}{c_k - c_{k-1}} \cdot (f(c_k) - f(c_{k-1})).$$

• Since the sequence $c_k$ is strictly increasing, the resulting function $f(x)$ is also strictly increasing.

• W.l.o.g., we assume $\mu(A) \geq \mu(B)$.

• There exist integers $k$ and $\ell$ for which $c_{k-1} < \mu(A) \leq c_{k+1}$ and $c_{\ell-1} < \mu(B) \leq c_{\ell}$; here, $k \geq \ell$ and $c_{\ell} \leq c_k$. 
19. Proof Finalized

• Hence, we have $\mu(A) \leq c_k$ and $\mu(B) \leq c_k$.

• By definition of $\Delta_k$, we therefore have $\mu(A \cup B) \leq \Delta_k$.

• By definition of $c_{k+1}$, this value is always greater than $\Delta_k$, thence we have $\mu(A \cup B) \leq c_{k+1}$.

• Since the function $f(x)$ is increasing, we get
  
  \[ \mu'(A \cup B) = f(\mu(A \cup B)) \leq f(c_{k+1}) = (1 + \varepsilon)^{(k+1)/2}. \]

• Here, $\max(\mu(A), \mu(B)) = \mu(A) > c_{k-1}$, so:
  
  \[ \max(\mu'(A), \mu'(B)) = \mu'(A) = f(\mu(A)) > f(c_{k-1}) = (1+\varepsilon)^{(k-1)/2}, \]

  i.e., $(1 + \varepsilon)^{(k-1)/2} < \max(\mu'(A), \mu'(B))$.

• Multiplying both sides by $1 + \varepsilon$, we get
  
  \[ (1 + \varepsilon)^{(k+1)/2} < (1 + \varepsilon) \cdot \max(\mu'(A), \mu'(B)). \]

• We already know that $\mu'(A \cup B) \leq (1 + \varepsilon)^{(k+1)/2}$.

• Thus, $\mu'(A \cup B) \leq (1 + \varepsilon) \cdot \max(\mu'(A), \mu'(B))$. Q.E.D.
20. Proof of the Auxiliary Result

- This proof is similar to the main result, the only difference is how we define $c_{k+1}$ and $c_{-(k+1)}$.
- By definition of an r-bounded measure, for each $i$, there exists a value $\Delta_{ki} > 0$ for which:
  
  $$\text{if } \mu_i(A) \leq c_k \text{ and } \mu_i(B) \leq c_k, \text{ then } \mu_i(A \cup B) \leq \Delta_{ki}.$$  

- We take $c_{k+1} = (1 + \varepsilon) \cdot \max(c_k, \Delta_{k1}, \ldots, \Delta_{kn})$.
- By definition of an r-bounded measure, for each $i$, there exists a value $\nu_{ki} > 0$ for which:
  
  $$\text{if } \mu_i(A) \leq \nu_{ki} \text{ and } \mu_i(B) \leq \nu_{ki}, \text{ then } \mu_i(A \cup B) \leq c_{-k}.$$  

- We take $c_{-(k+1)} = (1 - \varepsilon) \cdot \min(c_{-k}, \nu_{k1}, \ldots, \nu_{kn})$.
- The rest of the proof is the same.