Fuzzy Sets As Strongly Consistent Random Sets

Kittawit Autchariyapanitkul\textsuperscript{1}, Hung T. Nguyen\textsuperscript{2}, and Vladik Kreinovich\textsuperscript{3}

\textsuperscript{1}Faculty of Economics
Maejo University, Chiang Mai, Thailand, kittar3@hotmail.com

\textsuperscript{2}Dept. of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88003, USA, and Faculty of Economics
Chiang Mai University, Thailand, hunguyen@nmsu.edu

\textsuperscript{3}Dept. of Computer Science, University of Texas at El Paso
El Paso, Texas 79968, USA, vladik@utep.edu
1. Predictions Are Important

- One of the main applications of science and engineering:
  - is to predict future events, and
  - for engineering, to come up with designs and controls for which the future is the most beneficial.

- For example:
  - science predicts the position of the Moon in a few months, while
  - engineering predicts the position of the spaceship in a month, and
  - describes the best trajectory correction.
2. Predictions Are Important (cont-d)

- Some scientists say – correctly – that the main objective of science is to explain the world.
- But what does this mean in practical terms?
- How can we prove that a new physical theory explains the world better?
- By showing that it enables us to give more accurate predictions of future events.
- This is how General Relativity became accepted:
  - when experiments confirmed its prediction of
  - how much the light ray passing near the Sun will be distorted by the Sun’s gravitational field.
3. Perfect Knowledge is Rarely Available: Need for Set Uncertainty

- Usually, we have only partial knowledge.
- Thus, instead of a single future state, we have a set of future states.
- One way to predict the future state is to look for similar situations in the past.
- In the case of partial knowledge, we may have several different similar situations in the past.
- We can predict that the future situation will be similar to one of the corresponding outcomes.
4. From Set Uncertainty to Probabilistic Uncertainty

- When we have many similar situations,
  - we can determine not only which future states are possible,
  - but also how frequent are different future states.
- For each of the possible future states $s_1, \ldots, s_n$:
  - the observed frequency of this state
  - serves as a natural estimate for the probability $p_i$ of this state.
- Thus, in this case:
  - we know the set of possible states $s_1, \ldots, s_n$, and
  - we know the probabilities $p_1, \ldots, p_n$ of different possible states, probabilities adding up to 1: $\sum_{i=1}^{n} p_i = 1.$
5. From Probabilistic Uncertainty to Random Set Uncertainty

- Past observations were also only partial.
- We did not get a full knowledge of a state, we get a partial knowledge.
- So, each past observations $o_1, \ldots, o_n$ corresponds:
  - not to a single state, but
  - to the whole set $S_i$ of possible states.
- Example: a measuring instrument with accuracy 0.1 records:
  - the value 1.0 in 40% of the cases,
  - the value 1.1 in 20% of the cases, and
  - the value 1.2 in the remaining 20% of the cases.
6. Random Set Uncertainty (cont-d)

- This means that:
  - with probability 40%, we have values from the interval $S_1 = [0.9, 1.1]$;
  - with probability 20%, we have values from the interval $S_2 = [1.0, 1.2]$; and
  - with the remaining probability 20%, we have values from the interval $S_3 = [1.1, 1.3]$.

- A situation in which we have several sets with different probabilities is known as a random set.

- This is similar to how:
  - the situation when we have different numbers with different probabilities is a random number, and
  - the situation when we have different vectors with different probabilities is known as a random vector.
7. We Will Consider Finite Sets

- In practice:
  - because of the limits of measurement accuracy,
  - only finitely many different states are distinguishable.

- For example:
  - even if we can measure lengths from 0 to 1 m with accuracy of 1 micron,
  - we still have only a million possible values.

- Thus, in this paper, we will assume that our Universe of discourse $U$ is finite.

- Once a finite set $U$ is fixed, we can define a random set as a set of pairs $(S_i, p_i)$, where $S_i \subseteq U$, $p_i > 0$, and
\[ \sum_{i=1}^{n} p_i = 1. \]
8. Relation to Fuzzy Sets

- A fuzzy set, for each possible state $x \in U$, describes the degree $\mu(x)$ to which this state $x$ is possible.
- We can gauge this degree by the probability that $x$ is possible w.r.t. the corr. observation $S_i$ (i.e., $x \in S_i$):
  $$\mu(x) = \sum_{i:x \in S_i} p_i.$$  
- Every membership function can be thus interpreted: sort the values $\mu(x_1), \ldots, \mu(x_n)$ in a decreasing order:
  $$\mu(x(1)) \geq \mu(x(2)) \geq \ldots \geq \mu(x(n));$$  then:
  - $S_0 = \emptyset$ with prob. $p_0 = \mu(x(1))$;
  - $S_1 = \{x(1)\}$ with prob. $p_1 = \mu(x(1)) - \mu(x(2))$;
  - $S_k = \{x(1), \ldots, x(k)\}$, $p_k = \mu(x(k)) - \mu(x(k+1))$.
- Here, $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i:x(k) \in S_i} p_i = \mu(x(k))$ for all $k$. 

9. Relation to Fuzzy Sets (cont-d)

- For a normalized fuzzy set, for which $\max_k \mu(x_k) = 1$, there is no need for a weird empty set.

- There are other possible random sets that lead to the same fuzzy set $\mu$.

- As a result, we can interpret a fuzzy set $\mu(x)$ as an equivalence class of random sets.

- Namely, a fuzzy set is the class of all random sets for which, for every $x \in U$, we have $\sum_{i: x \in S_i} p_i = \mu(x)$.

- In general, a random set is nothing else but a mass distribution in the Dempster-Shafer approach, then

$$\mu(x) = \text{Pl}\{\{x\}\}.$$
10. Current Interpretation of Fuzzy Sets in Terms of Random Sets: Advantages and Limitations

• The above interpretation:
  – helps to teach fuzzy techniques to statisticians and
  – enables us to apply results about random sets to fuzzy techniques.

• The main problem with this interpretation is that it is too complicated.

• A random set is not an easy notion, and classes of random sets are even more complex.

• This complexity goes against the spirit of fuzzy sets, whose purpose was to be simple and intuitively clear.

• From this viewpoint, it is desirable to simplify this interpretation.
11. What We Do in This Paper

- We show that fuzzy sets can be interpreted
  - not as classes, but as
  - strongly consistent random sets (in some reasonable sense).

- This is not yet at the desired level of simplicity.

- However, this new interpretation is much simpler than the original one.

- It thus constitutes an important step towards the desired simplicity.
12. Notion of Consistency

- Often, different alternative are inconsistent and thus, different sets $S_i$ and $S_j$ are disjoint: $S_i \cap S_j = \emptyset$.

- Example: for accuracy 1.1, measurement results 1.0 and 1.3 are inconsistent: $[0.9, 1.1] \cap [1.2, 1.4] = \emptyset$.

- Sometimes, we have consistency: every two sets $S_i$ and $S_j$ have a non-empty intersection.

- This is true, e.g., for the random set that we used to represent a given fuzzy set.

- This is also true for the above example of three measurements 1.0, 1.1, and 1.2, with sets

  $$[0.9, 1.1], \quad [1.0, 1.2], \quad [1.1, 1.3]$$

- Let us require that the random set be consistent.
13. Need for Strong Consistency

- Sometimes, we learn an additional information, e.g., we learn that some alternative $x$ is not possible.
- In this case, a previously consistent random set may stop being consistent.
- Example: a random set with $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2, x_3\}$, and $p_1 = p_2 = 0.5$ is consistent:
  \[ S_1 \cap S_2 = \{x_2\} \neq \emptyset. \]
- However, if we learn that $x_2$ is not possible, it stops being consistent: for $S'_1 = \{x_1\}$ and $S'_2 = \{x_3\}$:
  \[ S'_1 \cap S'_2 = \emptyset. \]
- It is reasonable to require that the random set remain consistent when we learn additional information.
- We will call such random sets *strongly consistent*. 
14. How Does a Random Set Change When We Learn Additional Information

- Suppose that we had the Universe of discourse $U$.
- Then we learn that only some of the original alternatives are possible.
- Let $S \subseteq U$ denote the set of all possible alternatives; then: 
  \[ p(S'_i \mid S_i \text{ is possible}) = \frac{p_i}{\sum_{j: S_j \cap S \neq \emptyset} p_j}. \]
- In Dempster-Shafer terms, the denominator is equal to the plausibility $\text{Pl}(S)$ of the set $S$.
- Some of the sets may become equal, so we will have to combine their probabilities: 
  \[ p'(s) = \frac{\sum_{i: S_i \cap S = s} p_i}{\sum_{i: S_i \cap S \neq \emptyset} p_i}. \]
15. Definitions

- Let $U$ be a finite set; we will call this set the *Universe of discourse*.

- By a *random set*, we mean a pair $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$, where

  $$S_i \in U, \quad p_i > 0, \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1.$$ 

- A *fuzzy set* $\mu$ is a function from $U$ to $[0, 1]$.

- A fuzzy set $\mu(x)$ is called *normalized* if $\max_{x} \mu(x) = 1$.

- We say that a fuzzy set $\mu(x)$ is *consistent* with a random set $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ if for every $x \in U$:

  $$\mu(x) = \sum_{i:x \in S_i} p_i.$$
16. Definitions (cont-d)

- By a standard random set $S_\mu$ corresponding to the fuzzy set, we mean that following random set:
  
  - we sort the values $\mu(x)$ into the decreasing sequence $\mu(x(1)) \geq \ldots \geq \mu(x(n))$, and
  
  - we take $S_i = \{x(1), \ldots, x(i)\}$ with $p_i = \mu(x(i)) - \mu(x(i+1))$ for $i < n$ and $p_n = \mu(x(n))$.

- We say that a random set $((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ is consistent if $S_i \cap S_j \neq \emptyset$ for all $i$ and $j$.

- A set $S \subseteq U$ is consistent with a random set $S = ((S_1, \ldots, S_n), (p_1, \ldots, p_n))$ if $S \cap S_i \neq \emptyset$ for some $i$. 
17. Definitions (final)

- If $S$ is consistent with $\mathcal{S}$, we can define the restriction $S_S$ as follows:
  
  - it has non-empty sets $s$ of the type $S_i \cap S$
  
  - with probabilities $p'(s) = \sum_{i:S_i \cap S = s} p_i / \sum_{i:S_i \cap S \neq \emptyset} p_i$.

- We say that a random set is strongly consistent if all its restrictions are consistent.

- One can easily see that the standard random set corresponding to a fuzzy set is strongly consistent.

- Interestingly, as our result shows, the inverse is also true.
18. Main Result

• **Proposition.** For every normalized fuzzy set $\mu(x)$:
  
  – the only strongly consistent random set consistent with $\mu(x)$
  
  – is the standard random set $S_\mu$.

• One of the problems of the existing random-set interpretation of fuzzy sets is that in this interpretation:
  
  – each fuzzy set $\mu(x)$ is associated with
  
  – the whole *class* of random sets.

• If we restrict ourselves to strongly consistent $S$, then to each $\mu(x)$ there corresponds a *unique* random set.

• Thus, the interpretation of fuzzy sets as strongly consistent random sets is indeed simpler.
19. Proof: Main Idea

- Let us prove that if $S$ is strongly consistent, then for every two $S_i$ and $S_j$, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$.
- We will prove this by contradiction.
- Let us assume that for some $i$ and $j$, we have $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$, i.e., $S_i - S_j \neq \emptyset$ and $S_j - S_i \neq \emptyset$.
- Let us then consider the set $S = U - (S_i \cap S_j)$.
- This set $S$ is consistent with $S$, since

$$S \cap S_i = S_i - S_j \neq \emptyset \text{ and } S \cap S_j = S_j - S_i \neq \emptyset.$$  

- Thus, in the restriction $S_S$, we have elements

$$S'_i = S \cap S_i = S_i - S_j \text{ and } S'_j = S \cap S_j = S_j - S_i.$$ 

- One can easily see, however, that the sets $S'_i$ and $S'_j$ do not have any common elements.
20. Proof (cont-d)

• We have $S'_i \cap S'_j = \emptyset$.

• This contradicts to our assumption that $S$ is strongly consistent.

• The contradiction proves that the sets $S_i$ are indeed linearly ordered by inclusion.

• Thus, by a 2014 result by M. Daniel, $S$ is the standard random set associated with the given fuzzy set.
21. Acknowledgments

- This work was supported in part by the National Science Foundation grants:
  - HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and
  - DUE-0926721, and
- by an award from Prudential Foundation.