Isn’t Every Sufficiently Complex Logic Multi-Valued Already: Lindenbaum-Tarski Algebra and Fuzzy Logic Are Both Particular Cases of the Same Idea

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1. A Gap Between Fuzzy Logic and the Traditional 2-Valued Fuzzy Logic

- One of the main ideas behind fuzzy logic is that:
  - in contrast to the traditional 2-valued logic, in which every statement is either true or false,
  - in fuzzy logic, we allow intermediate degrees.

- In other words, fuzzy logic is an example of a multi-valued logic.

- This led to a misunderstanding between researchers in fuzzy and traditional logics.

- Fuzzy logic books claim that the 2-valued logic cannot describe intermediate degrees.

- On the other hand, 2-valued logicians criticize fuzzy logic for using “weird” intermediate degrees.
2. What We Do in This Paper

- We show that the mutual criticism is largely based on a misunderstanding.
- It *is* possible to describe intermediate degrees in the traditional 2-valued logic.
- However, such a representation is complicated.
- The main advantage of fuzzy techniques is that they provide a simply way of doing this.
- And simplicity is important for applications.
- We also show that the main ideas of fuzzy logic are consistent with the 2-valued foundations.
- Moreover, they naturally appear in these foundations if we try to adequately describe expert knowledge.
- We hope to help researchers from both communities to better understand each other.
3. Source of Multi-Valuedness in Traditional Logic: Gödel’s Theorem

- A naive understanding of the 2-valued logic assumes that every statement $S$ is either true or false.
- This is possible in simple situations.
- However, Gödel’s showed that this not possible for complex theories.
- Gödel analyzed arithmetic – statements obtained
  - from basic equalities and inequalities between polynomial expressions
  - by propositional connectives $\&$, $\lor$, $\neg$, and quantifiers over natural numbers.
- He showed that it is not possible to have a theory $T$ in which for every statement $S$, either $T \models S$ or $T \models \neg S$. 
4. We Have, in Effect, at Least Three Different Truth Values

- Due to Gödel’s theorem, there exist statements $S$ for which $T \not\models S$ and $T \not\models \neg S$. So:
  - while, legally speaking, the corresponding logic is 2-valued,
  - in reality, such a statement $S$ is neither true nor false.
- Thus, we have more than 2 possible truth values.
- At first glance, we have 3 truth values: “true”, “false”, and “unknown”.
- However, different “unknown” statements are not necessarily provably equivalent to each other.
- So, we may have more than 3 truth values.
5. How Many Truth Values Do We Actually Have

- It is reasonable to consider the following equivalence relation between statements $A$ and $B$:

$$\models (A \leftrightarrow B)$$

- Equivalence classes with respect to this relation can be viewed as the actual truth values.

- The set of all such equivalence classes is known as the *Lindenbaum-Tarski algebra*.

- Lindenbaum-Tarski algebra shows that any sufficiently complex logic is, in effect, multi-valued.

- However, this multi-valuedness is different from the multi-valuedness of fuzzy logic.

- We show that there is another close-to-fuzzy aspect of multi-valuedness of the traditional logic.
6. Need to Consider Several Theories

- In the previous section, we considered the case when we have a single theory $T$.
- G"odel’s theorem states that:
  - for every given theory $T$ that includes formal arithmetic,
  - there is a statement $S$ that can neither be proven nor disproven in this theory.
- This statement $S$ can neither be proven nor disproven based on the axioms of theory $T$.
- So, a natural idea is to consider additional reasonable axioms that we can add to $T$. 
7. Need to Consider Several Theories (cont-d)

- This is what happened in geometry with the V-th postulate $P$ – that
  - for every line $\ell$ in a plane and for every point $P$ outside this line,
  - there exists only one line $\ell'$ which passes through $P$ and is parallel to $\ell$.

- It turned out that neither $P$ nor $\neg P$ can be derived from all other (more intuitive) axioms of geometry.

- So, a natural solution is to explicitly add this statement as a new axiom.

- If we add its negation, we get Lobachevsky geometry – historically the first non-Euclidean geometry.
8. Need to Consider Several Theories (cont-d)

- A similar thing happened in set theory, with the Axiom of Choice and Continuum Hypothesis.
- They cannot be derived or rejected based on the other (more intuitive) axioms of set theory.
- Thus, they (or their negations) have to be explicitly added to the original theory.
- The new – extended – theory covers more statements than the original theory $T$.
- However, the same Gödel’s theory still applies to the new theory:
  - there are statements that
  - can neither be deduced nor rejected based on this new theory.
- Thus, we need to add one more axiom, etc.
9. We Have a Family of Theories

- So, instead of a single theory, it makes sense to consider a family of theories \( \{ T_\alpha \} \).

- In the above description, we end up with a family which is linearly ordered in the sense that:
  - for every two theories \( T_\alpha \) and \( T_\beta \),
  - either \( T_\alpha \models T_\beta \) or \( T_\beta \models T_\alpha \).

- However, it is possible that on some stage, different groups of researchers select two different axioms.

- In this case, we will have two theories which are not derivable from each other.

- Thus, we have a family of theories which is not linearly ordered.
10. How’s This Applicable to Expert Knowledge?

- We can select only the statements in which experts are 100% sure, and we get one possible theory.

- We can add statements $S$ for which the expert’s degree of confidence $d(S)$ exceeds a certain threshold $\alpha$:
  \[ \{ S : d(S) \geq \alpha \} . \]

- For different $\alpha$, we get different theories $T_\alpha$.

- For example, if we select $\alpha = 0.7$, then:
  - For every $x$ for which $\mu_{\text{small}}(x) \geq 0.7$, we consider $S(x)$ ("$x$ is small") to be true.
  - For all other objects $x$, we consider $S(x)$ to be false.

- Similarly, we only keep “if-then” rules for which the expert’s degree of confidence in these rules is $\geq 0.7$. 
11. Once We Have a Family of Theories, How Can We Describe the Truth of a Statement?

• If we have a single theory $T$, then for every $S$:
  – either $T \models S$, i.e., the statement $S$ is true in the theory $T$,
  – or $T \not\models \neg S$, i.e., $S$ is not true in the theory $T$.

• In general:
  – to describe whether a statement $S$ is true or not,
  – we should consider the values corresponding to all the theories $T_\alpha$.

• So, we should consider the whole set
  \[ \text{deg}(S) \overset{\text{def}}{=} \{ \alpha : T_\alpha \models S \}. \]

• This set is our degree of belief that $S$ is true – i.e., in effect, the truth value of the statement $S$. 
12. Logical Operations on the New Truth Values

- If a theory $T_\alpha$ implies both $S$ and $S'$, then this theory implies their conjunction $S \& S'$ as well.

- So, the truth value of the conjunction includes the intersection of truth value sets corresponding to $S$ and $S'$:
  \[
  \text{deg}(S \& S') \supseteq \text{deg}(S) \cap \text{deg}(S').
  \]

- Similarly, if a theory $T_\alpha$ implies either $S$ or $S'$, then this theory also implies the disjunction $S \lor S'$.

- Thus, the truth value of the disjunction includes the union of truth value sets corresponding to $S$ and $S'$:
  \[
  \text{deg}(S \lor S') \supseteq \text{deg}(S) \cup \text{deg}(S').
  \]
13. What Happens in the Simplest Case, When the Theories Are Linearly Ordered?

- If the theories $T_\alpha$ are linearly ordered, then, once $T_\alpha \models S$ and $T_\beta \models T_\alpha$, we also have $T_\beta \models S$.

- Thus, with every $T_\alpha$, the truth value $\text{deg}(S) = \{\alpha : T_\alpha \models S\}$ includes:
  - with each index $\alpha$,
  - the indices of all the stronger theories – i.e., all the theories $T_\beta$ for which $T_\beta \models T_\alpha$.

- In particular, for a finite family of theories, each degree is equal to $D_{\alpha_0} \overset{\text{def}}{=} \{\alpha : T_\alpha \models T_{\alpha_0}\}$ for some $\alpha_0$.

- In terms of the linear order $\alpha \leq \beta \iff T_\alpha \models T_\beta$, this degree takes the form $D_{\alpha_0} = \{\alpha : \alpha \leq \alpha_0\}$.

- We can thus view $\alpha_0$ as the degree of truth of the statement $S$: $\text{Deg}(S) \overset{\text{def}}{=} \alpha_0$. 
14. Linearly Ordered Case (cont-d)

• In case of expert knowledge, this means that we consider the smallest degree of confidence $d$ for which:
  – we can derive the statement $S$
  – if we allow all the expert’s statements whose degree of confidence is at least $d$.

• These sets $D_\alpha$ are also linearly ordered: one can easily show that $D_\alpha \subseteq D_\beta \iff \alpha \leq \beta$.

• The intersection of sets $D_\alpha$ and $D_\beta$ simply means that we consider the set $D_{\min(\alpha,\beta)}$.

• The union of sets $D_\alpha$ and $D_\beta$ simply means that we consider the set $D_{\max(\alpha,\beta)}$.

• Thus, the statements about $\&$ and $\lor$ take the form:

  $\text{Deg}(S \& S') \geq \min(\text{Deg}(S), \text{Deg}(S'))$;
  $\text{Deg}(S \lor S') \geq \max(\text{Deg}(S), \text{Deg}(S'))$. 
15. Relation to Fuzzy

- We have shown that:
  \[
  \text{Deg}(S \& S') \geq \min(\text{Deg}(S), \text{Deg}(S'));
  \]
  \[
  \text{Deg}(S \lor S') \geq \max(\text{Deg}(S), \text{Deg}(S')).
  \]

- The above formulas are very similar to the formulas of the fuzzy logic corresponding to min and max.

- The only difference is that we get $\geq$ instead of $=$.

- Thus, fuzzy logic ideas can be indeed naturally obtained in the classical 2-valued environment.

- They can be interpreted as a particular case of the same general idea as the Lindenbaum-Tarski algebra.
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