Which Material Design Is Possible Under Additive Manufacturing: A Fuzzy Approach

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1. Additive Manufacturing: Successes and Limitations

- Additive manufacturing – aka 3-D printing – is a promising way to generate complex material designs.
- It allows us to generate objects layer-by-layer, and thus, come up with very complex objects.
- While additive manufacturing has many successes, it is not a panacea.
- Current equipment for additive manufacturing only reproduces the desired design with a certain accuracy.
  - For simpler shapes, small deviations from the desired configuration do not affect their functionality.
  - However, for objects, with small elements, a small change in the configuration can ruin the result.
- Example: design of tiny blood vessels.
2. It Is Important to Estimate the Complexity of a Given Design

- An ideal blood vessel should have uniform width.
- A vessel with widely varying width impedes the blood flow.
- For such complex objects, we need more accurate equipment, whose use is very expensive.
- It is therefore desirable to be able to estimate the complexity of the design before the manufacturing, so that:
  - we would be able to see whether a given equipment can implement this design – and
  - if it can, whether this same design can be implemented by a cheaper equipment.

- The state-of-the-art empirical formula is based on the division of the design into several sections $i$.
- If we make sections sufficiently small, then in each section, we have at most two different materials.
- So there is no need to consider sections with three or more materials.
- Let us denote the fraction of this section which is filled with one of these materials by $v_i \in [0, 1]$.
- Then, the other material fills the fraction $1 - v_i$.
- In this case, the empirical formula for the complexity $C$ is $C = \sum_i v_i^n \cdot (1 - v_i)^\eta$, for some $\eta$. 
4. Open Problem

- When a formula does not have a theoretical justification, it is less reliable, because it is not clear
  - whether this dependence indeed follows from first principles — and thus, can be safely applied,
  - or is somewhat accidental and probably will not hold in other cases.

- In this paper, we provide a theoretical justification for this formula.

- In this justification, we use fuzzy logic ideas.
5. Commonsense Analysis of the Problem

- When we have only one material, i.e., when $v_i = 0$ or $v_i = 1$, there is no complexity: $c(0) = c(1) = 0$.
- When both materials are present, there is complexity, so we should have $c(v_i) > 0$ for $v_i \in (0, 1)$.
- Many physical ideas are based on the fact that:
  - every analytical function can be expanded in Taylor series, and,
  - as a good approximation, we can take the sum of the first few terms in this expansion.
- Let us consider $c(v_i) = c_0 + c_1 \cdot v_i + c_2 \cdot v_i^2 + c_3 \cdot v_i^3 + \ldots$
- 0-th order $c(v_i) = c_0$ is not enough: $c(0) = 0$ implies $c_0 = 0$ and $c(v_i) \equiv 0$, but $c(v_i) > 0$ for $v_i \in (0, 1)$.
- Linear approximation $c(v_i) = c_0 + c_1 \cdot v_i$ is also not good: $c(0) = c(1) = 0$ implies $c(v_i) = 0$ for all $v_i$. 
6. Commonsense Analysis and Fuzzy Logic

- So, we need quadratic terms: \( c(v_i) = c_0 + c_1 \cdot v_i + c_2 \cdot v_i^2 \).
- The conditions \( c(0) = c(1) = 0 \) imply that \( c(v_i) = c_1 \cdot (v_i - v_i^2) \), with \( c_1 > 0 \).
- We are interested in is relative complexity of designs.
- So, nothing will change if we divide all the complexity values by \( c_1 \) and take \( c(v_i) = v_i \cdot (1 - v_i) \).
- The section is complex if the first material is present and the second material is present.
- It is reasonable to take the proportion \( v_i \) as the degree to which this material is present in this section.
- Similarly, \( 1 - v_i \) can be taken as the degree to which the other material is present.
- If we use one of the simplest and widely used “and”-operations \( f_\& (a, b) = a \cdot b \), we get \( c = v_i \cdot (1 - v_i) \).
7. How Accurate Is Any Taylor Approximation?

- If addition of one more term drastically changes the situation, then:
  - our original approximation is rather crude, and
  - we should not trust the results of using this approximation too much.

- If the addition of one more term does not change the result, the original approximation was accurate.

- From this viewpoint, let us see what happens if we add a cubic term \( c(v_i) = c_0 + c_1 \cdot v_i + c_2 \cdot v_i^2 + c_3 \cdot v_i^3 \).

- Values \( v_i \) and \( 1 - v_i \) describe the same situation modulo re-naming, so, \( c(v_i) = c(1 - v_i) \).

- This implies \( c_3 = 0 \); so, an extra term doesn’t change much; thus, the quadratic approximate is accurate.
8. How to Combine Complexity of Sections into a Single Complexity Value?

- Let $f(C_1, C_2)$ be an overall complexity of a 2-section design with section complexities $C_1$ and $C_2$.

- The complexity does not depend on the order of the sections: $f(C_1, C_2) = f(C_2, C_1)$.

- The complexity of a 3-section design can be computed in two ways:
  - as combination of 1-2 and 3, then the complexity is $f(f(C_1, C_2), C_3)$,
  - or as combination as 1 and 2-3: $f(C_1, f(C_2, C_3))$.

- These values should coincide, so $f(C_1, C_2)$ must be associative.

- If we increase the complexity of one of the sections, the overall complexity increases, so $f$ is increasing.
Combining Complexities: Scale-Invariance

- Small changes in complexities $C_i$ should lead to small changes in the overall complexity; so $f$ is continuous.
- These are properties of “or”-operation (t-conorm) in fuzzy logic: namely, of Archimedean t-conorms.
- Thus, we can use the known classification of such t-conorms and conclude that
  \[ f(C_1, C_2) = g^{-1}(g(C_1) + g(C_2)) \] for some $g(C')$.
- This is equivalent to $g(C') = g(C_1) + g(C_2)$.
- As we have mentioned earlier, complexity is defined modulo a measuring unit $C \rightarrow \lambda \cdot C$.
- It is reasonable to require that the combination operation should not depend on this re-scaling, i.e., that
  \[ g(C') = g(C_1) + g(C_2) \implies g(\lambda \cdot C') = g(\lambda \cdot C_1) + g(\lambda \cdot C_2). \]
10. Our Result

- We have \( f(C_1, C_2) = g^{-1}(g(C_1) + g(C_2)) \).

- Thus, \( C = f(C_1, \ldots, c_N) \) if \( g(C) = \sum_{i=1}^{n} g(C_i) \).

- We prove that \( f \) is scale-invariant if and only if \( g(x) = \text{const} \cdot x^\eta \).

- Then, we have \( C = \left( \sum_{i=1}^{n} v_i^\eta \cdot (1 - v_i)^\eta \right)^{1/\eta} \).

- Our objective is to compare designs.

- It thus makes sense to consider simpler-to-describe re-scaled value \( \tilde{C} = C^n = \sum_{i=1}^{n} v_i^\eta \cdot (1 - v_i)^\eta \).

- This is exactly the empirical formula – we thus explained it.
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12. Proof

- Scale-invariance means that if we change $C_1 \rightarrow C_1' = C_1 + \Delta C_1$ and $C_2 \rightarrow C_2' = C_2 + \Delta C_2$, then
  \[ g(C_1') + g(C_2') = g(C_1) + g(C_2) \Rightarrow g(\lambda \cdot C_1') + g(\lambda \cdot C_2') = g(\lambda \cdot C_1) + g(\lambda \cdot C_2). \]
- $g(C_1') = g(C_1 + \Delta C_1) = g(C_1) + g'(C_1) \cdot \Delta C_1 + o(\Delta C_1)$.
- $g(C_2') = g(C_2) + g'(C_2) \cdot \Delta C_2 + o(\Delta C_2)$.
- Thus, $g(C_1') + g(C_2') = g(C_1) + g(C_2)$ implies
  \[ g'(C_1) \cdot \Delta C_1 + g'(C_2) \cdot \Delta C_2 + o(\Delta C_i) = 0, \text{ i.e.,} \]
  \[ \Delta C_2 = -\frac{g'(C_1)}{g'(C_2)} + o(\Delta C_i). \]
- Similarly, we have
  \[ g(\lambda \cdot C_1') = g(\lambda \cdot C_1) + g'(\lambda \cdot C_1) \cdot \lambda \cdot \Delta C_1 + o(\Delta C_1), \]
  \[ g(\lambda \cdot C_2') = g(\lambda \cdot C_2) + g'(\lambda \cdot C_2) \cdot \lambda \cdot \Delta C_2 + o(\Delta C_2). \]
13. Proof (cont-d)

• So, the condition \(g(\lambda \cdot C_1') + g(\lambda \cdot C_2') = g(\lambda \cdot C_1) + g(\lambda \cdot C_2)\) takes the form

\[
g'(\lambda \cdot C_1) \cdot \lambda \cdot \Delta C_1 + g'(\lambda \cdot C_2) \cdot \lambda \cdot \Delta C_2 + o(\Delta C_i) = 0.
\]

• Substituting the above expression for \(\Delta C_2\) in terms of \(\Delta C_1\) into this formula, we conclude that

\[
g'(\lambda \cdot C_1) \cdot \lambda \cdot \Delta C_1 - g'(\lambda \cdot C_2) \cdot \frac{g'(C_1)}{g'(C_2)} \cdot \lambda \cdot \Delta C_1 + o(\Delta C_i) = 0.
\]

• Dividing both sides by \(\lambda \cdot \Delta C_1\), we get

\[
g'(\lambda \cdot C_1) - g'(\lambda \cdot C_2) \cdot \frac{g'(C_1)}{g'(C_2)} + o(1) = 0.
\]

• When \(\Delta C_i \to 0\), we have \(g'(\lambda \cdot C_1) = g'(\lambda \cdot C_2) \cdot \frac{g'(C_1)}{g'(C_2)}\).

• Moving all the terms with \(C_1\) to one side and all the terms with \(C_2\) to another:

\[
\frac{g'(\lambda \cdot C_1)}{g'(C_1)} = \frac{g'(\lambda \cdot C_2)}{g'(C_2)}.
\]
14. Proof (cont-d)

- For all $C_1$ and $C_2$, we have $\frac{g'(\lambda \cdot C_1)}{g'(C_1)} = \frac{g'(\lambda \cdot C_2)}{g'(C_2)}$.

- Thus, the ratio $r \overset{\text{def}}{=} \frac{g'(\lambda \cdot C')}{g'(C)}$ does not depend on $C$ at all, it depends only in $\lambda$: $\frac{g'(\lambda \cdot C')}{g'(C)} = r(\lambda)$.

- So, $g'(\lambda \cdot C) = r(\lambda) \cdot g'(C)$.

- For every two values $\lambda_1$ and $\lambda_2$, we can:
  - apply this formula directly for $\lambda = \lambda_1 \cdot \lambda_2$, getting $g'(\lambda \cdot C') = r(\lambda) \cdot g'(C') = r(\lambda_1 \cdot \lambda_2) \cdot g'(C')$;
  - alternatively, we could apply it first for $\lambda_2$, getting $g'(\lambda_2 \cdot C') = r(\lambda_2) \cdot g'(C')$, and then for $\lambda_1$, getting $g'(\lambda \cdot C) = g'(\lambda_1 \cdot (\lambda_2 \cdot C')) = r(\lambda_1) \cdot g'(\lambda_2 \cdot C') = r(\lambda_1) \cdot (\lambda_2) \cdot g'(C')) = (r(\lambda_1) \cdot r(\lambda_2)) \cdot g'(C')$. 

15. Proof (cont-d)

• By comparing the two expressions for $g'(\lambda \cdot C)$, we get
  $$r(\lambda_1 \cdot \lambda_2) \cdot g'(C) = (r(\lambda_1) \cdot r(\lambda_2)) \cdot g'(C).$$

• Thus, $r(\lambda_1 \cdot \lambda_2) = r(\lambda_1) \cdot r(\lambda_2)$.

• It is known that every monotonic function $r(\lambda)$ satisfying the above equality has the form $r(\lambda) = \lambda^q$.

• Thus, $g'(\lambda \cdot C) = r(\lambda) \cdot g'(C)$ means $g'(\lambda \cdot C) = \lambda^q \cdot g'(C)$.

• For $\lambda = x$, $C = 1$, we get $g'(x) = c \cdot x^q$ ($c \overset{\text{def}}{=} g'(1)$).

• For $q = -1$, we get $g(x) = c \cdot \ln(x) + C_0$, but $g(0) = 0$.

• So, $q \neq -1$, and $g(x) = \frac{c}{q + 1} \cdot x^{q+1} + C_0$.

• $g(0) = 0$ implies $C_0 = 0$, so $g(x) = \text{const} \cdot x^{\eta}$, for $\eta \overset{\text{def}}{=} q + 1$.

• The result is proven.