Random Interval Arithmetic is Closer to Common Sense: An Observation

René Alt and Jean-Luc Lamotte
Laboratoire d’Informatique de Paris 6
Université Pierre et Marie Curie
4 Place Jussieu, 75252 cedex 05, Paris, France
rene.alt@upmc.fr, Jean-Luc.Lamotte@lip6.fr

Vladik Kreinovich
Department of Computer Science
University of Texas, El Paso, TX 79968
e-mail vladik@cs.utep.edu
1. Commonsense Arithmetic

- We have a bridge whose weight we know with an accuracy of 1 ton.
- On this bridge, we place a car whose weight we know with an accuracy of 5 kg.
- The accuracy of the overall weight is still 1 ton.
- This is what an engineer or a physicist would say.
- Related joke:
  - in 2000, a dinosaur was 14,000,000 years old;
  - so, in 2005, it must be 14,000,005 years old.
- What is desired: if $\Delta_a \gg \Delta_b$, and
  - we add “uncertainty approximately $\Delta_b$” to “uncertainty approximately $\Delta_a$”,
  - we should get “uncertainty approximately $\Delta_a$”.


2. Traditional Interval Arithmetic Does not Have the Desired Property

- A natural way of dealing with approximately known values is interval arithmetic.
- The value \( \tilde{a} \) with an accuracy \( \Delta_a \) is interpreted as an interval \( [\tilde{a} - \Delta_a, \tilde{a} + \Delta_a] \).
- Specifics:
  - we know \( \tilde{a} \) with uncertainty \( \Delta_a \);
  - we know \( \tilde{b} \) with uncertainty \( \Delta_b \);
  - then, \( a = [\tilde{a} - \Delta_a, \tilde{a} + \Delta_a] \), \( b = [\tilde{b} - \Delta_b, \tilde{b} + \Delta_b] \), and
  - so, the set of possible values of \( c = a + b \) is an interval
    \[
    c = a + b = [(\tilde{a} + \tilde{b}) - (\Delta_a + \Delta_b), (\tilde{a} + \tilde{b}) + (\Delta_a + \Delta_b)].
    \]
3. Traditional Interval Arithmetic Does not Have the Desired Property

- **Situation:**
  - we know $\tilde{a}$ with uncertainty $\Delta_a$;
  - we know $\tilde{b}$ with uncertainty $\Delta_b$;
  - we conclude that

  $$c \in [(\tilde{a} + \tilde{b}) - (\Delta_a + \Delta_b), (\tilde{a} + \tilde{b}) + (\Delta_a + \Delta_b)].$$

- **Interpretation:** we thus interpret this interval as

  “$\tilde{a} + \tilde{b}$ with uncertainty $\Delta_a + \Delta_b$”.

- **Conclusion:** if we know $a$ with uncertainty 1 ton, and we know $b$ with uncertainty 5 kg, then the resulting uncertainty in $a + b$ is 1.005 ton.

- **Problem:** how can we modify interval arithmetic?
4. Interval Arithmetic: Origins

- **Objective:** analyze how:
  - the uncertainty in input data, and
  - the round-off imprecision of computer operations
affect the results of the computations.

- **Traditional approach:** statistical techniques.

- **Problem:**
  - we must know the exact probability distributions of the input and round-off errors;
  - in practice, we *don’t know* these distributions.

- **What we do know:** upper bounds on the errors – i.e., *intervals* that contain them.

- *e.g.:* space navigation under uncertainty (NASA, 1950s).

- **Interval arithmetic** was developed.
5. Interval Arithmetic: Limitations

- **Problem:** producing the exact bounds on the inaccuracy of the output is often difficult (NP-hard).

- **Discussion:** the origin of interval techniques is in NASA-related problems that required high reliability.

- **Conclusion:** the emphasis in interval computations has always been on getting the *validated* results.

- Interval techniques produce estimates that are guaranteed to contain (enclose) the actual error.

- **Limitation:** it is often desirable,
  - in addition to guaranteed “overestimates”,
  - to produce a reasonable estimate of the size of the actual error,
  - an estimate that may be only valid with a certain probability.
6. Interval Arithmetic: Main Idea

- **Main idea:** we follow computations step by step.

- **Specifics:** for each intermediate computation step $z := x \odot y$,
  - once we have already computed the intervals $x = [x, \bar{x}]$ and $y = [y, \bar{y}]$ of possible values of $x$ and $y$,
  - we compute the interval for $z$.

- **Traditional interval arithmetic:** apply interval arithmetic operation to $x$ and $y$ corresponding to the worst case.

- **Example:** for addition,
  $$z = [x + y, \bar{x} + \bar{y}].$$
7. Random Interval Arithmetic

• **New idea (Vignes et al.) – motivation:**
  
  – depending on the relative monotonicity of the $x$ and $y$ relative to inputs,
  – the intervals $z$ can change from the worst-case situation to the best-case situation.

• **Best case arithmetic:** (a.k.a. dual or inner): e.g., for addition,

  \[ z = [\min(x + \bar{y}, \bar{x} + y), \max(x + \bar{y}, \bar{x} + y)] \]

• **Reasonable assumption:**
  
  – monotonicity in the same direction and
  – monotonicity in different directions

  are equally frequent.

• **Result:** on each step, we pick traditional or inner arithmetic with equal probability.
8. Random Interval Arithmetic Has the Desired Property

- *Example:* addition $c = a + b$.
- *Traditional arithmetic:* the half-width is:
  \[ \Delta^t_c = \Delta_a + \Delta_b. \]
- *Dual arithmetic:* $\Delta^d_c = \max(\Delta_a, \Delta_b) - \min(\Delta_a, \Delta_b)$.
- *Random interval arithmetic:* uses each operation with probability 50%.
- So, the average half-width of $c$ is $\Delta^r_c = \frac{\Delta^t_c + \Delta^d_c}{2}$.
- *Fact:* $\Delta^t_c = \Delta_a + \Delta_b = \max(\Delta_a, \Delta_b) + \min(\Delta_a, \Delta_b)$.
- *Conclusion:* $\Delta^r_c = \max(\Delta_a, \Delta_b)$.
- *Good news:* this is exactly the intuitive property that we have been trying to formalize.
9. What If We Add \( n \) Values?

- **Problem:**
  - we know each quantity \( a_i \) with an accuracy \( \Delta_i \);
  - what is the (expected value of) the accuracy in \( a = a_1 + \ldots + a_n \)?

- First, we add \( a_1 + a_2 \); the resulting accuracy is \( \max(\Delta_1, \Delta_2) \).

- To estimate the uncertainty of the next intermediate result \( (a_1 + a_2) + a_3 \), we take, as an estimate of the uncertainty in \( a_1 + a_2 \), the value \( \max(\Delta_1, \Delta_2) \).

- Then, the average uncertainty in \( (a_1 + a_2) + a_3 \) will be equal to
  \[
  \max(\max(\Delta_1, \Delta_2), \Delta_3) = \max(\Delta_1, \Delta_2, \Delta_3).
  \]

- Similarly, we conclude that the average uncertainty in \( a_1 + \ldots + a_n \) is equal to
  \[
  \max(\Delta_1, \ldots, \Delta_n).
  \]
10. Computing $f(a_1, a_2)$

- When $\Delta a_i \stackrel{\text{def}}{=} a_i - \tilde{a}_i \ll a_i$, we can safely linearize the expression for $f(a_1, a_2)$:

$$f(a_1, a_2) = f(\tilde{a}_1 + \Delta a_1, \tilde{a}_2 + \Delta a_2) =$$

$$f(\tilde{a}_1, \tilde{a}_2) + \frac{\partial f}{\partial a_1} \cdot \Delta a_1 + \frac{\partial f}{\partial a_2} \cdot \Delta a_2.$$ 

- So, when $\Delta a_i \in [\Delta_i, \Delta_i]$, the worst-case half-width in $a = f(a_1, a_2)$ is equal to

$$\Delta^t = \left| \frac{\partial f}{\partial a_1} \right| \cdot \Delta_1 + \left| \frac{\partial f}{\partial a_2} \right| \cdot \Delta_2.$$ 

- The result of applying dual interval arithmetic is

$$\Delta^d = \left| \frac{\partial f}{\partial a_1} \right| \cdot \Delta_1 - \left| \frac{\partial f}{\partial a_2} \right| \cdot \Delta_2.$$ 

- Thus, the average half-width – corresponding to random interval arithmetic – is equal to

$$\Delta^r = \max \left( \left| \frac{\partial f}{\partial a_1} \right| \cdot \Delta_1, \left| \frac{\partial f}{\partial a_2} \right| \cdot \Delta_2 \right).$$
11. **Computing** $f(a_1, \ldots, a_n)$

- Similarly, for $n > 2$ variables, we conclude that

$$\Delta^r = \max \left( \left| \frac{\partial f}{\partial a_1} \right| \Delta_1, \ldots, \left| \frac{\partial f}{\partial a_n} \right| \Delta_n \right).$$

- In interval computations, we estimate the range of a function over a box $[a_1, \overline{a}_1] \times \ldots \times [a_n, \overline{a}_n]$.

- If a box is not too narrow, the estimates are too wide.

- To improve the estimates, we:
  - bisect the box along one of the directions and
  - repeat the estimation for each of the two half-boxes.

- The optimal direction in a direction $a_i$ in which the product $\left| \frac{\partial f}{\partial a_i} \right| \Delta_i$ is the largest possible.

- The above value $\Delta^r$ is exactly the value of this maximum.
12. Relation with Fuzzy Logic

- **Our formula:**
  \[ \Delta_c = \max(\Delta_1, \Delta_b). \]

- **Fuzzy logic – objective:**
  - we know:
    * the degree of belief \( a = d(A) \) in a statement \( A \) and
    * the degree of belief \( b = d(B) \) in a statement \( B \),
  - we want to estimate the degree of belief \( c = d(C) \) in \( C \overset{\text{def}}{=} A \lor B \).

- In the most widely used (and most practically successful version) of fuzzy logic,
  \[ c = \max(a, b). \]
13. Acknowledgments

This work was supported in part:

- by NASA under cooperative agreement NCC5-209,
- by the NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, and
- by the NIH grant 3T34GM008048-20S1.