Two Etudes on Combining Probabilistic and Interval Uncertainty: Processing Correlations and Measuring Loss of Privacy

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1. Overview

- In many practical situations, there is a need to combine interval and probabilistic uncertainty.
- The need for such a combination leads to two types of problems:
  - how to process the given combined uncertainty, and
  - how to gauge the amount of uncertainty and – a related question – how to best decrease this uncertainty.
- In our research, we concentrate on these two types of problems.
- In this talk, we present two examples that illustrate how the corresponding problems can be solved.
2. General Problem of Data Processing under Uncertainty

- *Indirect measurements*: way to measure \( y \) that are difficult (or even impossible) to measure directly.

- *Idea*: \( y = f(x_1, \ldots, x_n) \)

  \[
  \begin{array}{c}
  \tilde{x}_1 \\
  \tilde{x}_2 \\
  \vdots \\
  \tilde{x}_n \\
  \end{array} \quad \rightarrow \quad f \quad \xrightarrow{\text{}} \quad \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)
  \]

- *Problem*: measurements are never 100\% accurate: \( \tilde{x}_i \neq x_i \) (\( \Delta x_i \neq 0 \)) hence

  \[
  \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n) \neq y = f(x_1, \ldots, y_n).
  \]

- *Question*: what are bounds on \( \Delta y \overset{\text{def}}{=} \tilde{y} - y \)?
3. Probabilistic and Interval Uncertainty

\[ \begin{align*}
\Delta x_1 & \quad f \\
\Delta x_2 \\
\vdots \\
\Delta x_n
\end{align*} \rightarrow \Delta y \]

- **Traditional approach:** we know probability distribution for \( \Delta x_i \) (usually Gaussian).
- **Where it comes from:** calibration using standard MI.
- **Problem:** calibration is not possible in:
  - fundamental science
  - manufacturing
- **Solution:** we know upper bounds \( \Delta_i \) on \(|\Delta x_i|\) hence
  \[ x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]. \]
4. **Interval Computations: A Problem**

Given: an algorithm $y = f(x_1, \ldots, x_n)$ and $n$ intervals $x_i = [x_i, \bar{x}_i]$.

Compute: the corresponding range of $y$:

$$[\underline{y}, \bar{y}] = \{f(x_1, \ldots, x_n) \mid x_1 \in [x_1, \bar{x}_1], \ldots, x_n \in [x_n, \bar{x}_n]\}.$$

Fact: NP-hard even for quadratic $f$.

Challenge: when are feasible algorithm possible?

Challenge: when computing $y = [\underline{y}, \bar{y}]$ is not feasible, find a good approximation $Y \supseteq y$. 
5. **Interval Arithmetic: Foundations of Interval Techniques**

- **Problem:** compute the range
  \[ [y, \overline{y}] = \{ f(x_1, \ldots, x_n) \mid x_1 \in [x_1, \overline{x}_1], \ldots, x_n \in [x_n, \overline{x}_n] \}. \]

- **Interval arithmetic:** for arithmetic operations \( f(x_1, x_2) \) (and for elementary functions), we have explicit formulas for the range.

- **Examples:** when \( x_1 \in \mathbf{x}_1 = [x_1, \overline{x}_1] \) and \( x_2 \in \mathbf{x}_2 = [x_2, \overline{x}_2] \), then:
  - The range \( \mathbf{x}_1 + \mathbf{x}_2 \) for \( x_1 + x_2 \) is \( [x_1 + x_2, \overline{x}_1 + \overline{x}_2] \).
  - The range \( \mathbf{x}_1 - \mathbf{x}_2 \) for \( x_1 - x_2 \) is \( [x_1 - \overline{x}_2, \overline{x}_1 - x_2] \).
  - The range \( \mathbf{x}_1 \cdot \mathbf{x}_2 \) for \( x_1 \cdot x_2 \) is \( [y, \overline{y}] \), where
    \[
    y = \min(x_1 \cdot x_2, x_1 \cdot \overline{x}_2, \overline{x}_1 \cdot x_2, \overline{x}_1 \cdot \overline{x}_2);
    \overline{y} = \max(x_1 \cdot x_2, x_1 \cdot \overline{x}_2, \overline{x}_1 \cdot x_2, \overline{x}_1 \cdot \overline{x}_2).
    \]
  - The range \( 1/\mathbf{x}_1 \) for \( 1/x_1 \) is \( [1/\overline{x}_1, 1/x_1] \) (if \( 0 \notin \mathbf{x}_1 \)).
6. Straightforward Interval Computations: Example

- **Example:** \( f(x) = (x - 2) \cdot (x + 2), \ x \in [1, 2]. \)

- How will the computer compute it?
  - \( r_1 := x - 2; \)
  - \( r_2 := x + 2; \)
  - \( r_3 := r_1 \cdot r_2. \)

- **Main idea:** perform the same operations, but with *intervals* instead of *numbers*:
  - \( r_1 := [1, 2] - [2, 2] = [-1, 0]; \)
  - \( r_2 := [1, 2] + [2, 2] = [3, 4]; \)
  - \( r_3 := [-1, 0] \cdot [3, 4] = [-4, 0]. \)

- **Actual range:** \( f(x) = [-3, 0]. \)

- **Comment:** this is just a toy example, there are more efficient ways of computing an enclosure \( Y \supseteq y. \)
7. Combining Interval and Probabilistic Uncertainty

- **Situation:** in some cases, in addition to the bounds on each variables, we have partial information about its probability distribution.
- **Problem:** there are many ways to represent a probability distribution.
- **Idea:** look for an objective.
- **Objective:** make decisions $E_x[u(x, a)] \rightarrow \max a$.
- **Analysis:** for smooth $u(x)$, we have
  \[
u(x) = u(x_0) + (x-x_0) \cdot u'(x_0) + \frac{1}{2} \cdot (x-x_0)^2 \cdot u''(x_0) + \ldots
  \]
  so
  \[
  E[u(x)] = u(x_0) + E[x-x_0] \cdot u'(x_0) + \frac{1}{2} \cdot E[(x-x_0)^2] \cdot u''(x_0) + \ldots
  \]
- **Conclusion:** we must know moments to estimate $E[u]$. 
8. Extension of Interval Arithmetic to Probabilistic Case: Successes

- **Easy cases:** $+, -, \text{ product of independent } x_i$.

- **Example of a non-trivial case:** multiplication $y = x_1 \cdot x_2$, when we have no information about the correlation:

  $\begin{align*}
  E &= \max(p_1 + p_2 - 1, 0) \cdot \overline{x}_1 \cdot \overline{x}_2 + \min(p_1, 1 - p_2) \cdot \bar{x}_1 \cdot \bar{x}_2 + \\
  &\quad \min(1 - p_1, p_2) \cdot x_1 \cdot \bar{x}_2 + \max(1 - p_1 - p_2, 0) \cdot \bar{x}_1 \cdot x_2; \\
  \overline{E} &= \min(p_1, p_2) \cdot \overline{x}_1 \cdot \overline{x}_2 + \max(p_1 - p_2, 0) \cdot \bar{x}_1 \cdot \bar{x}_2 + \\
  &\quad \max(p_2 - p_1, 0) \cdot x_1 \cdot \bar{x}_2 + \min(1 - p_1, 1 - p_2) \cdot \bar{x}_1 \cdot x_2,
  \end{align*}$

  where $p_i \overset{\text{def}}{=} (E_i - x_i)/(\overline{x}_i - x_i)$. 

9. First Result

- **Problem**: the above expression is computationally complicated.
- **New result**: new, equivalent, more computationally efficient expressions for $E$ and $\overline{E}$:

$$E = E_1 \cdot E_2 - \min ((E_1 - x_1) \cdot (E_2 - x_2), (x_1 - E_1) \cdot (x_2 - E_2));$$

$$\overline{E} = E_1 \cdot E_2 + \min ((E_1 - x_1) \cdot (x_2 - E_2), (x_1 - E_1) \cdot (E_2 - x_2)).$$
10. Taking Correlation into Account: A Problem

- **Fact:** the range of $E[x_1 \cdot x_2]$ depends on the ranges of $E[x_i]$ and on the correlation between the $x_i$.

- **Previously covered:**
  
  - case when $x_1$ and $x_2$ are independent, and
  
  - case when we have no information about their correlation.

- **Practical situation:** sometimes, we know the interval $[\rho, \bar{\rho}]$ of possible values of the correlation $\rho$:
  
  $$\rho(x_1, x_2) \overset{\text{def}}{=} \frac{E[x_1 \cdot x_2] - E_1 \cdot E_2}{\sigma_1 \cdot \sigma_2}.$$  

- **Question:** what is the resulting range of $E[x_1 \cdot x_2]$?
11. Taking Correlation into Account: First Result

• Given:
  • $[x_1, \bar{x}_1]$ and $[x_2, \bar{x}_2]$ are given intervals,
  • $E_1 \in [x_1, \bar{x}_1]$ and $E_2 \in [x_1, \bar{x}_1]$ are given numbers, and
  • $\rho$ is a given number.

• Find: the range $[\underline{E}, \bar{E}]$ of possible values $E[x_1 \cdot x_2]$ for all possible distributions for which:
  • $x_1$ is located in $[x_1, \bar{x}_1]$, and $x_2$ is located in $[x_2, \bar{x}_2]$;
  • $E[x_1] = E_1$, and $E[x_2] = E_2$; and
  • $\rho[x_1, x_2] = \rho$.

• Solution:
  • for $\rho \geq 0$: $[\underline{E}, \bar{E}] = [E_1 \cdot E_2, E_1 \cdot E_2 + \rho \cdot \sigma]$;
  • for $\rho \leq 0$: $[\underline{E}, \bar{E}] = [E_1 \cdot E_2 + \rho \cdot \sigma, E_1 \cdot E_2]$. 
12. Taking Correlation into Account: Second Result

- **Given:**
  - \([x_1, \bar{x}_1]\) and \([x_2, \bar{x}_2]\) are given intervals;
  - \(E_1 \in [x_1, \bar{x}_1]\) and \(E_2 \in [x_1, \bar{x}_1]\) are given numbers;
  - \([\rho, \bar{\rho}]\) is a given interval.

- **Find:** the range \([\underline{E}, \overline{E}]\) of possible values \(E[x_1 \cdot x_2]\) for all possible distributions for which:
  - \(x_1\) is located in \([x_1, \bar{x}_1]\), and \(x_2\) is located in \([x_2, \bar{x}_2]\);
  - \(E[x_1] = E_1\), and \(E[x_2] = E_2\); and
  - \(\rho[x_1, x_2] \in [\rho, \bar{\rho}]\).

- **Solution:**
  - for \(0 \leq \rho\): \([\underline{E}, \overline{E}] = [E_1 \cdot E_2, E_1 \cdot E_2 + \bar{\rho} \cdot \sigma]\);
  - for \(\bar{\rho} \leq 0\): \([\underline{E}, \overline{E}] = [E_1 \cdot E_2 + \rho \cdot \sigma, E_1 \cdot E_2]\);
  - for \(\rho \leq 0 \leq \bar{\rho}\): \([\underline{E}, \overline{E}] = [E_1 \cdot E_2 + \rho \cdot \sigma, E_1 \cdot E_2 + \bar{\rho} \cdot \sigma]\).
13. Second Problem: How to Measure Loss of Privacy

- **Measuring loss of privacy is important:** to compare different privacy protection schemes.
- **Natural idea:** gauge the loss of privacy by the resulting worst-case financial loss.
- **Example:** the effect of a person’s blood pressure $x$ on this person’s insurance payments:
  - let $f(x)$ be average medical expenses for a person with blood pressure $x$; let $\alpha$ be investment profit;
  - in case of privacy, the insurance payments are
    \[
    r = (1 + \alpha) \cdot E[f(x)];
    \]
  - if a person’s blood pressure is revealed as $x_0$, with $f(x_0) > E[f(x)]$, then the payments are higher:
    \[
    r_0 = (1 + \alpha) \cdot f(x_0) > r = (1 + \alpha) \cdot E[f(x)].
    \]
14. Loss of Privacy: Main Result

- **Situation:** we knew that $x \in [L, U]$, now we learned that $x \in [l, u] \subseteq [L, U]$.

- **Description:** we knew that $P \in \mathcal{P}$ (all distributions located on $[L, U]$), now we know that $P \in \mathcal{Q}$ (all distributions located on $[l, u]$).

- **Definition:** let $M > 0$ be a real number. The *amount of privacy* $A(\mathcal{P})$ related to $\mathcal{P}$ is the largest possible value of the difference $F(x_0) - \int \rho(x) \cdot F(x) \, dx$ over:
  - all possible values $x_0$,
  - all possible probability distributions $\rho \in \mathcal{P}$, and
  - all possible f-s $F(x)$ for which $|F'(x)| \leq M$ for all $x$.

- **Result:** the *relative loss of privacy* $\frac{A(\mathcal{P}) - A(\mathcal{Q})}{A(\mathcal{P})}$ is equal to $1 - \frac{u - l}{U - L}$. 
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