How to Avoid Gerrymandering: A New Algorithmic Solution

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1. Re-Districting: A Practical Problem

- *The notion of electoral districts:* in the USA and in other countries, voting is done by electoral districts.
- *First objective:* equal representation.
- *How:* all voting districts of the same level (federal, state, city, etc.) contain same number of voters.
- *Problem:* in time, demography changes – some districts lose voters, some gain them.
- *Solution:* re-districting.
- *Main objective of re-districting:* best represent geographic regions.
- *Additional objective:* represent rural and city areas, minorities, border areas, flooded areas, etc.
- *Open problem:* how to take all these factors into account?

2. Gerrymandering: A Problem

- *How re-districting is done:* in most states, it is voted upon by the legislature.
- *Drawback:* in the next elections, the representation may be unfairly biased towards a party in power.
- *Phenomenon behind bias:* for parties A and B, A votes in B-majority district are "lost" – this district votes for B anyway.
- *Gerrymandering – how:* a party in power (A)
  - divides all the A voters into A-majority districts, and
  - attaches, to each A-district, many B-voters (w/out violating A-majority).
- *Result:* many B votes are lost, while no A votes are lost.

3. Gerrymandering: Example

- *Example:* 18 towns on similar size:
  - $A_1, \ldots, A_{10}$ vote for A,
  - $B_1, \ldots, B_8$ vote for B.
- *Objective:* form 6 electoral districts.
- *Fair representation:* A gets $\dfrac{10}{18} \cdot 6 = \dfrac{10}{3} = 3.33\ldots$ votes – i.e., 3 or 4.
- *Example of gerrymandering:* divide 18 towns into the following 6 districts:
  - $\{A_1, A_2, B_1\}$, $\{A_3, A_4, B_2\}$, $\{A_5, A_6, B_3\}$, $\{A_7, A_8, B_4\}$, and $\{A_9, A_{10}, B_5\}$ vote for A;
  - only 1 district out of 6, $\{B_6, B_7, B_8\}$ votes for B.
- *Result:* A gets 5 votes out of 6.

4. How to Avoid Gerrymandering: Known Approaches

- *Typical idea:* limit the "weirdness" of the district shapes.
- *Example:* make them as round as possible, with the smallest possible length of the separation lines.
- *Limitations:*
  - these approaches only take into account geographical closeness;
  - they may not give adequate representation to minorities or rural population.
- *Remaining problem:* take into account
  - not only geographical closeness and differences,
  - but also other types of closeness and difference.

5. Toward Formulation of the Problem in Precise Mathematical Terms

- *Representing voters:* by parameters $x_1, \ldots, x_n$:
  - geographic location (x_i are geographic coordinates),
  - income,
  - rural vs. urban status,
  - number of children, etc.
- *Representatives* are described by $v_1, \ldots, v_n$.
- *Utility approach:* in decision making, preferences are represented by utility functions:

$$u = u(x_1 - v_1, \ldots, x_n - v_n).$$

- *Different characteristics are usually independent:* it is known that then $u(d_1, \ldots, d_n) = u_1(d_1) + \ldots + u_n(d_n)$.
- *Differences are small:* representatives are close to the voters, so differences $d_i \stackrel{\text{def}}{=} r_i - v_i$ are small.

6. Toward Formulation of the Problem in Precise Mathematical Terms (cont-d)

- *Taylor expansion:* since differences $d_i = r_i - v_i$ are small,

$$u_i(d_i) = u_i(0) + u_i'(0) \cdot d_i + \frac{u''(0)}{2} \cdot d_i^2 + \ldots$$

- *Analysis:* the largest possible utility is when a representative is a perfect match, i.e., $d_i = 0$.

- *Conclusion:* $u_i(d_i) \approx u_i(0) - w_i \cdot d_i^2$ and

$$u(d_1, \ldots, d_n) = \sum_{i=1}^{n} u_i(d_i) = \sum_{i=1}^{n} u_i(0) - \sum_{i=1}^{n} w_i \cdot d_i^2.$$

- Thus, maximizing utility is equivalent to minimizing the *disutility*

$$\rho(x, v) = \sum_{i=1}^{n} w_i \cdot (r_i - v_i)^2.$$

7. Resulting Formulation of the Problem

- We have:
  - an integer n (number of characteristics),
  - positive real numbers $w_1, \ldots, w_n$ (weights);
  - n-dimensional vectors $x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$ $(1 \le k \le N)$ describing voters;
  - number c of voting districts.
- *Objective:*
  - subdivide N voters into c groups $D_1, \ldots, D_c$, and
  - select a vector $v(1), \ldots, v(c)$ within each group
  - so as to minimize the overall disutility

$$\sum_{j=1}^{c} \sum_{k \in D_j} \rho(x^{(k)}, v(j)).$$

8. Relation to Clustering

- *Informal objective:* a voter is closer to other voters from his/her district than to voters from other districts.
- *Relation to clustering:* such groups are called clusters.
- *Iterative clustering:* we start with some representations $v(1), \ldots, v(c)$, then repeat the following 2 steps:
  - each voter $x^{(k)}$ is assigned to the group D_j for which the disutility $\rho(x^{(k)}, v(j))$ is the smallest;
  - after that, for each group D_j, we re-calculate $v(j)$ as the average of all $x^{(k)}$ $(k \in D_j)$.
- *Limitation of this approach:* unequal clusters.
- *Example:* a population consisting of a (larger) city and a (smaller) rural area.
- *Resulting clusters:* a larger all-city cluster and a smaller all-rural cluster.

9. Towards a More Adequate Solution

- *Main idea:* groups $D_1, \ldots, D_c$ must have equal size $\dfrac{N}{c}$.
- *Technical challenge:* when we have the representatives $v(j)$, how can we get districts?
- *Technical result:* in the optimal districting, the division between D_i and D_j is determined by a threshold t_{ij} for the ratio $r_{ij}(x) \stackrel{\text{def}}{=} \dfrac{\rho(x, v(i))}{\rho(x, v(j))}$:
  - if $r_{ij}(x) < t_{ij}$, then $x \in D_i$;
  - if $r_{ij}(x) > t_{ij}$, then $x \in D_j$.
- *Conclusion:* for some weights α_j, a voter x is assigned to the class D_j for which $\alpha_j \cdot \rho(x^{(k)}, v(j))$ is the largest.
- *Remaining problem:* find the weights α_j for which the resulting districts D_j are of equal size.

10. Towards an Algorithm

- *We start:* with $\alpha_1^{(0)} = \ldots = \alpha_c^{(0)} = 1$.
- *Start iteration:* with some values $\alpha_1^{(p)}, \ldots, \alpha_c^{(p)}$.
- *Main part of the iteration:* for each j, we find β_j for which there are exactly N/c points x for which

$$\beta_j \cdot \rho(x, v(j)) \le \alpha_k^{(p)} \cdot \rho(x, v(k))$$

  for all $k \ne j$.
- *How to find β_j:* bisection
  - if we get $< N/c$ points, we decrease β_j;
  - if we get $> N/c$ points, we increase β_j.
- *Observation:* if we multiply all the values α_j by the same constant, we get the same classes.
- *Last part of the iteration:* find α_j for which $\dfrac{\alpha_j}{\alpha_k} \approx \dfrac{\beta_j}{\alpha_k^{(p)}}$ for all $j \ne k$.

11. Analysis of the Auxiliary Problem

- *Problem – reminder:* find α_j for which $\dfrac{\alpha_j}{\alpha_k} \approx \dfrac{\beta_j}{\alpha_k^{(p)}}$ for all $j \neq k$.
- *Difficulty:* this problem is non-linear in unknowns α_j.
- *Idea:* turn to logarithms $A_j \stackrel{\text{def}}{=} \ln(\alpha_j)$, $B_j \stackrel{\text{def}}{=} \ln(\beta_j)$, and $A_j^{(p)} \stackrel{\text{def}}{=} \ln(\alpha_j^{(p)})$.
- *New problem:* find A_j for which

$$A_j - A_k \approx B_j - A_k^{(p)}.$$

- *Least Squares solution:*

$$A_j = \frac{1}{2} \cdot (B_j + A_j^{(p)}) + \text{const}.$$

- *Resulting value of α_j:*

$$\alpha_j = \sqrt{\beta_j \cdot \alpha_j^{(p)}}.$$

12. Example

- *Situation:* uniform population distribution on $[0,1]$.
- *Starting reps:* $v(1)=0$, $v(2)=0.5$, and $v(3)=1.0$.
- *Simple clustering:* $D_1=[0,0.25]$, $D_2=[0.25,0.75]$, $D_3=[0.75,1]$.
- *Problem:* D_2 is twice larger than D_1 or D_3.
- *1st iteration:* find β_1 for which $\beta_1\cdot x^2\le(0.5-x)^2$ for exactly $1/3$ of points.
- *Computing β_j:* $\beta_1=0.25$, similarly $\beta_2=4$, $\beta_3=0.25$.
- *Computing α_j:* $\alpha_1^{(2)}=\sqrt{\beta_1\cdot\alpha_1^{(1)}}=\sqrt{0.25\cdot 1}=0.5$; similarly, $\alpha_2^{(2)}=\sqrt{\beta_2\cdot\alpha_2^{(1)}}=\sqrt{4\cdot 1}=2$, $\alpha_3^{(2)}=0.5$.
- *Result:* $1/3$ of voters in the first district, for which $\alpha_1\cdot\rho(x,v(1))\to\min$, and $1/3$ each in D_2 and D_3.

13. Resulting Iterative Algorithm

- *Start:* representations $v(1), \ldots, v(c)$ (e.g., representatives of the existing districts).
- *Main idea:* iterations of the 2-step process:
  - subdivide voters into c equal groups corr. to $v(j)$;
  - re-calculate $v(j)$ as the average of the j-th group.
- *1st step starts* with an iterative process:
  - start with weights $\alpha_1^{(1)} = \ldots = \alpha_c^{(1)} = 1$;
  - use bisection to find β_j for which there are exactly N/c points $x^{(k)}$ for which, for all $l \neq j$:
  $$\beta_j \cdot \rho(x^{(k)}, v(j)) \leq \alpha^{(p)} \cdot \rho(x^{(k)}, v(l));$$
  - compute $\alpha_j^{(p+1)} = \sqrt{\beta_j \cdot \alpha_j^{(p)}}$.
- *1st step ends:* assign each voter x to to the group D_j for which $\alpha_j \cdot \rho(x^{(k)}, v(j))$ is the smallest.

14. Conclusion

- *What we propose:* a new algorithm for dividing an area into voting districts.
- *New features:* we take take into account not only geographic closeness, but also common interests of voters.
- *Necessary input:* weights w_i of different factors:
  - geographic location (x_i are geographic coordinates),
  - income,
  - rural vs. urban status,
  - number of children,
  - etc.

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