Asymmetric Information Measures: How to Extract Knowledge From an Expert so That the Expert’s Effort Is Minimal

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1. How Knowledge Is Extracted Now

- *Knowledge acquisition:* we ask experts questions, and put the answers into the computer system.

- *Problem:* it is a very time-consuming and therefore expensive task.

- *Objective:* minimize the effort of an expert.

- *Related problem:* how do we estimate this effort?

- *Reasonable idea:* number of binary ("yes"-"no") questions.

- *Resulting strategy:* binary search.

- *Idea:* we choose a question for which the answer is "yes" for exactly half of the remaining alternatives.

- *Property:* we need $\log_2(N)$ questions to select one of $N$ alternatives.
2. Experts Are Usually More Comfortable with “Yes” Answers

- **In practice:** most people feel more comfortable answering “yes” than “no”.

- **Fact:** the expert’s time is valuable.

- **Consequence:** an expert is usually called after competent people tried to solve the problem.

- **Expected situation:** the expert mostly confirms their preliminary solutions.

- **Consequence:** most expert’s answers are “yes”.

- **Binary search case:** half of the answers are “no”s.

- **Meaning:** half of the previous decisions were wrong.

- **Expert’s conclusion:** no competent people tried this problem – so his/her valuable time was wasted.
3. Experts Are Usually More Comfortable with “Yes” Answers (cont-d)

- **Situation**: a knowledge engineer interviews the expert.

- **First alternative**: most answers are “yes”; meaning:
  - the knowledge engineer already has some preliminary knowledge of the area, and
  - he/she is appropriately asking these questions to improve this knowledge.

- **Binary search**: half of the answers are “no” (same as for random questions); interpretation:
  - the knowledge engineer did not bother to get preliminary knowledge;
  - the highly skilled expert is inappropriately used to answer questions
  - which could be answered by consulting a textbook.
4. **Problem**

- *Reminder:* experts prefer “yes” answers.

- *Additional phenomenon:*
  - the larger the number of negative answers,
  - the more discomfort the expert will experience, and
  - the larger effort he will have to make to continue this interview.

- *Previous objective:* minimize the total number of questions.

- *More appropriate objective:* minimize the effort of an expert.

- *How to describe the effort:* assign more weight to “no” answers than to “yes” ones.

- *What we do:* find a search procedure which attains this objective.
5. How to Describe Different Search Procedures

• Let $S$ be the set of $N$ alternatives.

• We denote “yes” as 1, “no” as 0, so each sequence of answers $\omega$ is a binary sequence.

• To describe a search procedure, we must have:
  
  – the set $\Omega$ of possible answer sequences $\omega$, and
  
  – a mapping $A$ which maps each $\omega \in \Omega$ to the set $A(\omega)$ of all alternatives which are consistent with $\omega$.

• Formally: $A(\Lambda) = S$, and for every $\omega \in \Omega$:
  
  • if $|A(\omega)| = 1$, then no extension of $\omega$ belongs to $\Omega$;
  
  • otherwise, $\omega_0 \in \Omega$, $\omega_1 \in \Omega$, and we have

  $$A(\omega) = A(\omega_0) \cup A(\omega_1), \quad A(\omega_0) \cap A(\omega_1) = \emptyset,$$

  $$A(\omega_0) \neq \emptyset, \quad A(\omega_1) \neq \emptyset.$$
6. How to Gauge Different Search Procedures

- Let $P = (\Omega, A)$ be a search procedure.
- Let $W_0$ be the cost of “no” answer, and $W_1 < W_0$ be the cost of the “yes” answer.
- For $a \in \Omega$, let $\omega(a, P) = \omega_1\omega_2 \ldots \omega_k$ denote the sequence of answers which leads to $a$.
- The cost $W(\omega(a, P))$ of finding $a$ is defined as $W(\omega(a, P)) = W(\omega_1\omega_2 \ldots \omega_k) = W_{\omega_1} + W_{\omega_2} + \ldots + W_{\omega_k}$.
- The effort of a procedure is defined as the largest of its costs: $E(P) = \max_{a \in S} W(\omega(a, P))$.
- Objective: find a procedure $P_{\text{opt}}$ with the smallest possible effort: $E(P_{\text{opt}}) = T(N) \overset{\text{def}}{=} \min_P E(P)$. 
7. Example 1: Binary Search (Optimal for $W_0 = W_1$)

- **Situation:** a doctor chooses between $N = 4$ possible analgetics:
  - aspirin ($as$),
  - acetaminophen ($ac$),
  - ibuprofen ($ib$), and
  - valium ($va$).

- **Binary search:**
  
  \[
  \begin{array}{c}
  A(\Lambda) \\
  \text{↙} \ \text{↘} \\
  A(0) \quad A(1) \\
  \text{↙} \ \text{↘} \ \text{↙} \ \text{↘} \\
  A(00) \quad A(01) \quad A(10) \quad A(11) \\
  as \quad ac \quad ib \quad va
  \end{array}
  \]
8. Example 2: A Search Procedure Which Is Better Than Binary ($W_0 > W_1$)

- When $W_1 = 1$ and $W_0 = 3$, the effort of the binary search is 6.

- We can decrease the effort to 5 by applying the following alternative procedure:

```
  A(Λ)
  ↙  ↘
A(0)  A(1)
as  ↙  ↘
A(10) A(11)
  ↙  ↘
A(110) A(111)
  ↙  ↘
ib   va
```
9. Description of the Optimal Search Procedure

- **Auxiliary result:**

  \[ T(N) = \min_{0 < N_+ < N} \{ \max\{W_1 + T(N_+), W_0 + T(N - N_+)\} \}. \]

- **Conclusion:** we can consequently compute \( T(1), T(2), \ldots, T(N) \) in time \( N \cdot O(N) = O(N^2) \).

- **Notation:** let \( N_+(N) \) be the value where the minimum is attained.

- **Optimal procedure:** for each sequence \( \omega \) with \( n \overset{\text{def}}{=} |A(\omega)| > 1 \):
  
  - we assign \( N_+(n) \) values to the “yes” case \( A(\omega_1) \);
  
  - we assign the remaining \( n - N_+(n) \) values to the “no” case \( A(\omega_0) \).
10. **Example:** $N = 4$, $W_0 = 3$, and $W_1 = 1$

- We take $T(1) = 0$. Then,
  
  $$T(2) = \min_{0 < N_+ < 2} \left\{ \max\{1 + T(N_+), 3 + T(2 - N_+)\} \right\} = \max\{1 + T(1), 3 + T(1)\} = \max\{1, 3\} = 3,$$
  with $N_+(2) = 1$.

- $T(3) = 4$, with min attained for $N_+(3) = 2$.

- $T(4) = 5$, with min attained for $N_+(4) = 3$.

- **Optimal procedure:**
  
  - since $N_+(4) = 3$, we divide 4 elements $A(\Lambda)$ into a 3-element set $A(1)$ and a 1-element set $A(0)$;
  
  - since $N_+(3) = 2$, we divide 3 elements $A(1)$ into a 2-element set $A(11)$ and a 1-element set $A(10)$;
  
  - since $N_+(2) = 1$, we divide 2 elements $A(10)$ into a 1-element set $A(101)$ and a 1-element set $A(100)$.

- **Observation:** this is the procedure from Example 2.
11. Asymptotically Optimal Search Procedure

- **We described**: optimal search procedure:

\[ E(P_{\text{opt}}) = T(N) \overset{\text{def}}{=} \min_P E(P). \]

- **Property**: \( P_{\text{opt}} \) takes time \( \approx N^2 \).

- **Problem**: for large \( N \), time \( N^2 \) is too large.

- **Alternative**: asymptotically optimal procedure, with \( E(P_a) \leq T(N) + C \) for some constant \( C > 0 \).

- **Asymptotically optimal search procedure**:
  - find \( \alpha \) such that \( \alpha + \alpha^w = 1 \), where \( w \overset{\text{def}}{=} W_0/W_1 \);
  - for each \( \omega \) with \( n \overset{\text{def}}{=} |A(\omega)| > 1 \), assign \( \lfloor \alpha \cdot n \rfloor \) values to the “yes” case \( A(\omega_1) \);
  - assign the remaining values to the “no” case \( A(\omega_0) \).
12. **Examples**

- **Reminder:** we find $\alpha$ such that $\alpha + \alpha^w = 1$, where $w \stackrel{\text{def}}{=} W_0/W_1$.

- **Example 1:**
  - description: $W_0 = W_1 = 1$, so $w = 1$;
  - equation: $\alpha + \alpha = 1$;
  - solution: $\alpha = 0.5$;
  - resulting algorithm: binary search.

- **Example 2:**
  - description: $W_0 = 2$, $W_1 = 1$, so $w = 2$;
  - equation: $\alpha + \alpha^2 = 1$;
  - solution: $\alpha = \frac{\sqrt{5} - 1}{2} \approx 0.618$ is the golden ratio;
  - resulting algorithm: asymmetric search.
13. Average Case vs. Worst Case

• **Reminder:** we gauged each procedure $P$ by its worst-case effort $E(P) = \max_{a \in S} W(\omega(a, P))$.

• **Alternative:** use the average-case effort

$$E^a(P) \overset{\text{def}}{=} \frac{1}{N} \cdot \sum_{a \in S} W(\omega(a, P)).$$

• **Problem:** find $P_{opt}$ for which

$$E^a(P_{opt}) = T^a(N) \overset{\text{def}}{=} \min_P E^a(P).$$

• **Auxiliary result:** $T^a(N) = \min_{0 < N_+ < N} \left\{ \frac{N_+}{N} \cdot (W_1 + T^a(N_+)) + \frac{N - N_+}{N} \cdot (W_0 + T^a(N - N_+)) \right\}.

• **Notation:** let $N_+^a(N)$ be the value where the minimum is attained.
14. **Average Case: Optimal and Asymptotically Optimal Search Procedures**

- **Optimal procedure:** for each $\omega$ with $n \overset{\text{def}}{=} |A(\omega)| > 1$,
  - assign $N_A^+(n)$ values to the “yes” case $A(\omega1)$;
  - assign the remaining values to the “no” case $A(\omega0)$.

- **Asymptotically optimal procedure:**
  - find $K^a \geq 0$ and $\alpha^a$ for which:
    \[
    \alpha^a \cdot W_1 + (1 - \alpha^a) \cdot W_0 + K^a \cdot (\alpha^a \cdot \log_2(\alpha^a) + (1 - \alpha^a) \cdot \log_2(1 - \alpha^a)) = 0;
    \]
    \[
    W_0 - W_1 = K^a \cdot (\log_2(\alpha^a) - \log_2(1 - \alpha^a));
    \]
  - for each $\omega$ with $n \overset{\text{def}}{=} |A(\omega)| > 1$, assign $\lfloor \alpha^a \cdot n \rfloor$ values to the “yes” case $A(\omega1)$;
  - assign the remaining values to the “no” case $A(\omega0)$.
15. Observation

- **Reminder:** one of the equations is

$$K^a = \frac{\alpha^a \cdot W_1 + (1 - \alpha^a) \cdot W_0}{-\alpha^a \cdot \log_2(\alpha^a) - (1 - \alpha^a) \cdot \log_2(1 - \alpha^a)}.$$

- **Reminder:** $\alpha^a$ and $1 - \alpha^a$ are the probabilities of the “yes” and “no” answers.

- **First conclusion:** the numerator $\alpha^a \cdot W_1 + (1 - \alpha^a) \cdot W_0$ is the average effort.

- **Second conclusion:** the denominator $-\alpha^a \cdot \log_2(\alpha^a) - (1 - \alpha^a) \cdot \log_2(1 - \alpha^a)$ is the entropy of the probability distribution.

- **General conclusion:**

$$K^a = \frac{\text{average effort}}{\text{entropy of the probability distribution}}.$$
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