Application-Motivated Combinations of Fuzzy, Interval, and Probability Approaches, with Application to Geoinformatics, Bioinformatics, and Engineering

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Interval computations website:
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1. General Problem of Data Processing under Uncertainty

- **Indirect measurements**: way to measure $y$ that are difficult (or even impossible) to measure directly.

- **Idea**: $y = f(x_1, \ldots, x_n)$

- **Problem**: measurements are never 100% accurate: $\tilde{x}_i \neq x_i$ ($\Delta x_i \neq 0$) hence

\[ \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n) \neq y = f(x_1, \ldots, x_n). \]

What are bounds on $\Delta y \overset{\text{def}}{=} \tilde{y} - y$?
2. Probabilistic and Interval Uncertainty

- Traditional approach: we know probability distribution for $\Delta x_i$ (usually Gaussian).
- Where it comes from: calibration using standard MI.
- Problem: calibration is not possible in:
  - fundamental science
  - manufacturing
- Solution: we know upper bounds $\Delta_i$ on $|\Delta x_i|$ hence
  $$x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i].$$
3. Interval Computations: A Problem

- **Given**: an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) intervals \( x_i = [x_i, \bar{x}_i] \).

- **Compute**: the corresponding range of \( y \):
  \[
  [\underline{y}, \overline{y}] = \{ f(x_1, \ldots, x_n) \mid x_1 \in [x_1, \bar{x}_1], \ldots, x_n \in [x_n, \bar{x}_n] \}.
  \]

- **Fact**: NP-hard even for quadratic \( f \).

- **Challenge**: when are feasible algorithm possible?

- **Challenge**: when computing \( y = [\underline{y}, \overline{y}] \) is not feasible, find a good approximation \( Y \supseteq y \).
4. Alternative Approach: Maximum Entropy

- *Situation:* in many practical applications, it is very difficult to come up with the probabilities.

- *Traditional engineering approach:* use probabilistic techniques.

- *Problem:* many different probability distributions are consistent with the same observations.

- *Solution:* select one of these distributions – e.g., the one with the largest entropy.

- *Example – single variable:* if all we know is that \( x \in [\underline{x}, \overline{x}] \), then MaxEnt leads to a uniform distribution on \([\underline{x}, \overline{x}]\).

- *Example – multiple variables:* different variables are independently distributed.
5. Limitations of Maximum Entropy Approach

- **Example:** simplest algorithm $y = x_1 + \ldots + x_n$.

- **Measurement errors:** $\Delta x_i \in [-\Delta, \Delta]$.

- **Analysis:** $\Delta y = \Delta x_1 + \ldots + \Delta x_n$.

- **Worst case situation:** $\Delta y = n \cdot \Delta$.

- **Maximum Entropy approach:** due to Central Limit Theorem, $\Delta y$ is $\approx$ normal, with $\sigma = \Delta \cdot \frac{\sqrt{n}}{\sqrt{3}}$.

- **Why this may be inadequate:** we get $\Delta \sim \sqrt{n}$, but due to correlation, it is possible that $\Delta = n \cdot \Delta \sim n \gg \sqrt{n}$.

- **Conclusion:** using a single distribution can be very misleading, especially if we want guaranteed results.

- **Examples:** high-risk application areas such as space exploration or nuclear engineering.
6. **Interval Arithmetic: Foundations of Interval Techniques**

- **Problem:** compute the range

\[ [y, \overline{y}] = \{ f(x_1, \ldots, x_n) | x_1 \in [x_1, \overline{x_1}], \ldots, x_n \in [x_n, \overline{x_n}] \} \]

- **Interval arithmetic:** for arithmetic operations \( f(x_1, x_2) \) (and for elementary functions), we have explicit formulas for the range.

- **Examples:** when \( x_1 \in x_1 = [x_1, \overline{x_1}] \) and \( x_2 \in x_2 = [x_2, \overline{x_2}] \), then:
  - The range \( x_1 + x_2 \) for \( x_1 + x_2 \) is \([x_1 + x_2, \overline{x_1} + \overline{x_2}]\).
  - The range \( x_1 - x_2 \) for \( x_1 - x_2 \) is \([x_1 - \overline{x_2}, \overline{x_1} - x_2]\).
  - The range \( x_1 \cdot x_2 \) for \( x_1 \cdot x_2 \) is \([y, \overline{y}]\), where
    \[
    y = \min(x_1 \cdot x_2, x_1 \cdot \overline{x_2}, \overline{x_1} \cdot x_2, \overline{x_1} \cdot \overline{x_2});
    \]
    \[
    \overline{y} = \max(x_1 \cdot x_2, x_1 \cdot \overline{x_2}, \overline{x_1} \cdot x_2, \overline{x_1} \cdot \overline{x_2}).
    \]
- The range \( 1/x_1 \) for \( 1/x_1 \) is \([1/\overline{x_1}, 1/x_1]\) (if \( 0 \not\in x_1 \)).
7. **Straightforward Interval Computations: Example**

- **Example:** \( f(x) = (x - 2) \cdot (x + 2), \ x \in [1, 2]. \)

- How will the computer compute it?
  
  - \( r_1 := x - 2; \)
  - \( r_2 := x + 2; \)
  - \( r_3 := r_1 \cdot r_2. \)

- **Main idea:** perform the same operations, but with intervals instead of numbers:
  
  - \( r_1 := [1, 2] - [2, 2] = [-1, 0]; \)
  - \( r_2 := [1, 2] + [2, 2] = [3, 4]; \)
  - \( r_3 := [-1, 0] \cdot [3, 4] = [-4, 0]. \)

- **Actual range:** \( f(x) = [-3, 0]. \)

- **Comment:** this is just a toy example, there are more efficient ways of computing an enclosure \( Y \supseteq y. \)
8. Case Study: Chip Design

- *Chip design*: one of the main objectives is to decrease the clock cycle.
- *Current approach*: uses worst-case (interval) techniques.
- *Problem*: the probability of the worst-case values is usually very small.
- *Result*: estimates are over-conservative – unnecessary over-design and under-performance of circuits.
- *Difficulty*: we only have *partial* information about the corresponding probability distributions.
- *Objective*: produce estimates valid for all distributions which are consistent with this information.
- *What we do*: provide such estimates for the clock time.

- **Objective:** estimate the clock cycle on the design stage.

- The clock cycle of a chip is constrained by the maximum path delay over all the circuit paths

  \[ D \overset{\text{def}}{=} \max(D_1, \ldots, D_N). \]

- The path delay \( D_i \) along the \( i \)-th path is the sum of the delays corresponding to the gates and wires along this path.

- Each of these delays, in turn, depends on several factors such as:
  - the variation caused by the current design practices,
  - environmental design characteristics (e.g., variations in temperature and in supply voltage), etc.
10. Traditional (Interval) Approach to Estimating the Clock Cycle

- **Traditional approach:** assume that each factor takes the worst possible value.
- **Result:** time delay when all the factors are at their worst.
- **Problem:**
  - different factors are usually independent;
  - combination of worst cases is improbable.
- **Computational result:** current estimates are 30% above the observed clock time.
- **Practical result:** the clock time is set too high – chips are over-designed and under-performing.
11. Robust Statistical Methods Are Needed

- **Ideal case:** we know probability distributions.
- **Solution:** Monte-Carlo simulations.
- **In practice:** we only have *partial* information about the distributions of some of the parameters; usually:
  - the mean, and
  - some characteristic of the deviation from the mean
    - e.g., the interval that is guaranteed to contain possible values of this parameter.
- **Possible approach:** Monte-Carlo with several possible distributions.
- **Problem:** no guarantee that the result is a valid bound for all possible distributions.
- **Objective:** provide *robust* bounds, i.e., bounds that work for all possible distributions.
Towards a Mathematical Formulation of the Problem

• **General case:** each gate delay $d$ depends on the difference $x_1, \ldots, x_n$ between the actual and the nominal values of the parameters.

• **Main assumption:** these differences are usually small.

• Each path delay $D_i$ is the sum of gate delays.

• **Conclusion:** $D_i$ is a linear function: $D_i = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j$

  for some $a_i$ and $a_{ij}$.

• The desired maximum delay $D = \max_i D_i$ has the form

\[
D = F(x_1, \ldots, x_n) \overset{\text{def}}{=} \max_i \left( a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j \right).
\]
13. Towards a Mathematical Formulation of the Problem (cont-d)

- **Known:** maxima of linear function are exactly convex functions:
  \[ F(\alpha \cdot x + (1 - \alpha) \cdot y) \leq \alpha \cdot F(x) + (1 - \alpha) \cdot F(y) \]
  for all \( x, y \) and for all \( \alpha \in [0, 1] \);

- **We know:** factors \( x_i \) are independent;
  - we know distribution of some of the factors;
  - for others, we know ranges \([x_j, \bar{x}_j]\) and means \(E_j\).

- **Given:** a convex function \( F \geq 0 \) and a number \( \varepsilon > 0 \).

- **Objective:** find the smallest \( y_0 \) s.t. for all possible distributions, we have \( y \leq y_0 \) with the probability \( \geq 1 - \varepsilon \).
14. Additional Property: Dependency is Non-Degenerate

- **Fact:** sometimes, we learn additional information about one of the factors $x_j$.

- **Example:** we learn that $x_j$ actually belongs to a proper subinterval of the original interval $[x_j, \bar{x}_j]$.

- **Consequence:** the class $\mathcal{P}$ of possible distributions is replaced with $\mathcal{P}' \subset \mathcal{P}$.

- **Result:** the new value $y'_0$ can only decrease: $y'_0 \leq y_0$.

- **Fact:** if $x_j$ is irrelevant for $y$, then $y'_0 = y_0$.

- **Assumption:** irrelevant variables been weeded out.

- **Formalization:** if we narrow down one of the intervals $[x_j, \bar{x}_j]$, the resulting value $y_0$ decreases: $y'_0 < y_0$. 
15. Formulation of the Problem

**GIVEN:**
- \( n, \ k \leq n, \ \varepsilon > 0; \)
- a convex function \( y = F(x_1, \ldots, x_n) \geq 0; \)
- \( n - k \) cdfs \( F_j(x), \ k + 1 \leq j \leq n; \)
- intervals \( x_1, \ldots, x_k, \) values \( E_1, \ldots, E_k, \)

**TAKE:** all joint probability distributions on \( R^n \) for which:
- all \( x_i \) are independent,
- \( x_j \in x_j, \ E[x_j] = E_j \) for \( j \leq k, \) and
- \( x_j \) have distribution \( F_j(x) \) for \( j > k. \)

**FIND:** the smallest \( y_0 \) s.t. for all such distributions,
\( F(x_1, \ldots, x_n) \leq y_0 \) with probability \( \geq 1 - \varepsilon. \)

**WHEN:** the problem is *non-degenerate* – if we narrow down one of the intervals \( x_j, \) \( y_0 \) decreases.
16. Main Result and How We Can Use It

- **Result:** $y_0$ is attained when for each $j$ from 1 to $k$,
  - $x_j = x_j$ with probability $p_j \overset{\text{def}}{=} \frac{x_j - E_j}{x_j - x_j}$, and
  - $x_j = \bar{x}_j$ with probability $\bar{p}_j \overset{\text{def}}{=} \frac{E_j - x_j}{x_j - x_j}$.

- **Algorithm:**
  - simulate these distributions for $x_j$, $j < k$;
  - simulate known distributions for $j > k$;
  - use the simulated values $x_j^{(s)}$ to find
    $$y^{(s)} = F(x_1^{(s)}, \ldots, x_n^{(s)});$$
  - sort $N$ values $y^{(s)}$: $y(1) \leq y(2) \leq \cdots \leq y(N_i)$;
  - take $y(N_i \cdot (1 - \varepsilon))$ as $y_0$. 

17. Comment about Monte-Carlo Techniques

- **Traditional belief:** Monte-Carlo methods are inferior to analytical:
  - they are approximate;
  - they require large computation time;
  - simulations for *several* distributions, may mis-calculate the (desired) maximum over *all* distributions.

- **We proved:** the value corresponding to the selected distributions indeed provide the desired maximum value $y_0$.

- **General comment:**
  - justified Monte-Carlo methods often lead to *faster* computations than analytical techniques;
  - example: multi-D integration – where Monte-Carlo methods were originally invented.
18. Comment about Non-Linear Terms

- **Reminder:** in the above formula \( D_i = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j \), we ignored quadratic and higher order terms in the dependence of each path time \( D_i \) on parameters \( x_j \).

- **In reality:** we may need to take into account some quadratic terms.

- **Idea behind possible solution:** it is known that the max \( D = \max_i D_i \) of convex functions \( D_i \) is convex.

- **Condition when this idea works:** when each dependence \( D_i(x_1, \ldots, x_k, \ldots) \) is still convex.

- **Solution:** in this case,
  - the function function \( D \) is still convex,
  - hence, our algorithm will work.
19. Conclusions

- **Problem of chip design:** decrease the clock cycle.

- **How this problem is solved now:** by using worst-case (interval) techniques.

- **Limitations of this solution:** the probability of the worst-case values is usually very small.

- **Consequence:** estimates are over-conservative, hence over-design and under-performance of circuits.

- **Objective:** find the clock time as \( y_0 \) s.t. for the actual delay \( y \), we have \( \text{Prob}(y > y_0) \leq \varepsilon \) for given \( \varepsilon > 0 \).

- **Difficulty:** we only have partial information about the corresponding distributions.

- **What we have described:** a general technique that allows us, in particular, to compute \( y_0 \).
20. Combining Interval and Probabilistic Uncertainty: General Case

- **Problem**: there are many ways to represent a probability distribution.
- **Idea**: look for an objective.
- **Objective**: make decisions $E_x[u(x, a)] \rightarrow \max_a$
- **Case 1**: smooth $u(x)$.
  - **Analysis**: we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \ldots$
  - **Conclusion**: we must know moments to estimate $E[u]$.
- **Case of uncertainty**: interval bounds on moments.
- **Case 2**: threshold-type $u(x)$.
  - **Conclusion**: we need cdf $F(x) = \text{Prob}(\xi \leq x)$.
  - **Case of uncertainty**: p-box $[F(x), \bar{F}(x)]$.
21. Extension of Interval Arithmetic to Probabilistic Case: Successes

- **General solution:** parse to elementary operations +, −, ·, 1/x, max, min.

- Explicit formulas for arithmetic operations known for intervals, for p-boxes \( F(x) = [\underline{F}(x), \overline{F}(x)] \), for intervals + 1st moments \( E_i \overset{\text{def}}{=} E[x_i] \):

\[
\begin{array}{c}
\underbrace{x_1, \underline{E}_1} \\
\underbrace{x_2, \underline{E}_2} \\
\vdots \\
\underbrace{x_n, \underline{E}_n} \\
\end{array} \quad \overset{f}{\longrightarrow} \quad \underbrace{y, \underline{E}}
\]

\( E_i \) represents the expected value of the random variable \( x_i \).
22. Successes (cont-d)

- **Easy cases:** +, −, product of independent $x_i$.

- **Example of a non-trivial case:** multiplication $y = x_1 \cdot x_2$, when we have no information about the correlation:

  - $\bar{E} = \max(p_1 + p_2 - 1, 0) \cdot \overline{x_1} \cdot \overline{x_2} + \min(p_1, 1 - p_2) \cdot \overline{x_1} \cdot x_2 + \min(1 - p_1, p_2) \cdot x_1 \cdot \overline{x_2} + \max(1 - p_1 - p_2, 0) \cdot x_1 \cdot x_2$;

  - $\bar{E} = \min(p_1, p_2) \cdot \overline{x_1} \cdot \overline{x_2} + \max(p_1 - p_2, 0) \cdot \overline{x_1} \cdot x_2 + \max(p_2 - p_1, 0) \cdot x_1 \cdot \overline{x_2} + \min(1 - p_1, 1 - p_2) \cdot \overline{x_1} \cdot \overline{x_2}$,

where $p_i \defeq (E_i - x_i) / (\overline{x_i} - x_i)$. 

23. **Challenges**

- intervals + 2nd moments:
  
  \[
  x_1, E_1, V_1 \quad \rightarrow \quad f \quad \rightarrow \quad y, E, V
  \]
  
  \[
  x_2, E_2, V_2 \quad \rightarrow \quad \vdots \quad \rightarrow \quad E, F(x)
  \]

- moments + p-boxes; e.g.:
  
  \[
  E_1, F_1(x) \quad \rightarrow \quad f \quad \rightarrow \quad E, F(x)
  \]
  
  \[
  E_2, F_2(x) \quad \rightarrow \quad \vdots \quad \rightarrow \quad E_n, F_n(x)
  \]
24. Case Study: Bioinformatics

- **Practical problem:** find genetic difference between cancer cells and healthy cells.

- **Ideal case:** we directly measure concentration $c$ of the gene in cancer cells and $h$ in healthy cells.

- **In reality:** difficult to separate.

- **Solution:** we measure $y_i \approx x_i \cdot c + (1 - x_i) \cdot h$, where $x_i$ is the percentage of cancer cells in $i$-th sample.

- **Equivalent form:** $a \cdot x_i + h \approx y_i$, where $a \overset{\text{def}}{=} c - h$. 
25. Case Study: Bioinformatics (cont-d)

- **If we know** \( x_i \) **exactly**: Least Squares Method
  \[
  \sum_{i=1}^{n} (a \cdot x_i + h - y_i)^2 \rightarrow \min, \text{ hence } a = \frac{C(x, y)}{V(x)} \text{ and } h = E(y) - a \cdot E(x), \text{ where } E(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i,
  \]
  \[
  V(x) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x))^2,
  \]
  \[
  C(x, y) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x)) \cdot (y_i - E(y)).
  \]

- **Interval uncertainty**: experts manually count \( x_i \), and only provide interval bounds \( x_i \), e.g., \( x_i \in [0.7, 0.8] \).

- **Problem**: find the range of \( a \) and \( h \) corresponding to all possible values \( x_i \in [\underline{x}_i, \overline{x}_i] \).
26. General Problem

- **General problem:**
  
  - we know intervals $\mathbf{x}_1 = [x_1, \overline{x}_1], \ldots, \mathbf{x}_n = [x_n, \overline{x}_n],$
  
  - compute the range of $E(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$, population variance $V = \frac{1}{n} \sum_{i=1}^{n} (x_i - E(x))^2$, etc.

- **Difficulty:** NP-hard even for variance.

- **Known:**
  
  - efficient algorithms for $V$,
  
  - efficient algorithms for $\overline{V}$ and $C(x, y)$ for reasonable situations.

- **Bioinformatics case:** find intervals for $C(x, y)$ and for $V(x)$ and divide.
27. Case Study: Detecting Outliers

- In many application areas, it is important to detect outliers, i.e., unusual, abnormal values.
- In medicine, unusual values may indicate disease.
- In geophysics, abnormal values may indicate a mineral deposit (or an erroneous measurement result).
- In structural integrity testing, abnormal values may indicate faults in a structure.
- Traditional engineering approach: a new measurement result \( x \) is classified as an outlier if \( x \not\in [L, U] \), where 
  \[ L \equiv E - k_0 \cdot \sigma, \quad U \equiv E + k_0 \cdot \sigma, \]
  and \( k_0 > 1 \) is pre-selected.
- Comment: most frequently, \( k_0 = 2, 3, \) or 6.
28. Outlier Detection Under Interval Uncertainty: A Problem

- In some practical situations, we only have intervals $x_i = [x_i, \overline{x}_i]$.
- Different $x_i \in x_i$ lead to different intervals $[L, U]$.
- A possible outlier: outside some $k_0$-sigma interval.
- Example: structural integrity – not to miss a fault.
- A guaranteed outlier: outside all $k_0$-sigma intervals.
- Example: before a surgery, we want to make sure that there is a micro-calcification.
- A value $x$ is a possible outlier if $x \notin [L, U]$.
- A value $x$ is a guaranteed outlier if $x \notin [L, \overline{U}]$.
- Conclusion: to detect outliers, we must know the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$. 
29. Outlier Detection Under Interval Uncertainty: A Solution

- **We need:** to detect outliers, we must compute the ranges of \( L = E - k_0 \cdot \sigma \) and \( U = E + k_0 \cdot \sigma \).

- **We know:** how to compute the ranges \( E \) and \( [\sigma, \bar{\sigma}] \) for \( E \) and \( \sigma \).

- **Possibility:** use interval computations to conclude that \( L \in E - k_0 \cdot [\sigma, \bar{\sigma}] \) and \( L \in E + k_0 \cdot [\sigma, \bar{\sigma}] \).

- **Problem:** the resulting intervals for \( L \) and \( U \) are wider than the actual ranges.

- **Reason:** \( E \) and \( \sigma \) use the same inputs \( x_1, \ldots, x_n \) and are hence not independent from each other.

- **Practical consequence:** we miss some outliers.

- **Desirable:** compute exact ranges for \( L \) and \( U \).

- **Application:** detecting outliers in gravity measurements.
30. **Fuzzy Computations: A Problem**

Given: an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) fuzzy numbers \( \mu_i(x_i) \).

Compute: \( \mu(y) = \max_{x_1, \ldots, x_n : f(x_1, \ldots, x_n) = y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)) \).

Motivation: \( y \) is a possible value of \( Y \leftrightarrow \exists x_1, \ldots, x_n \) s.t. each \( x_i \) is a possible value of \( X_i \) and \( f(x_1, \ldots, x_n) = y \).

Details: “and” is min, \( \exists \) (“or”) is max, hence

\[
\mu(y) = \max_{x_1, \ldots, x_n} \min(\mu_1(x_1), \ldots, \mu_n(x_n), t(f(x_1, \ldots, x_n) = y)),
\]

where \( t(\text{true}) = 1 \) and \( t(\text{false}) = 0 \).
31. Fuzzy Computations: Reduction to Interval Computations

- **Problem (reminder):**
  - Given: an algorithm \( y = f(x_1, \ldots, x_n) \) and \( n \) fuzzy numbers \( X_i \) described by membership functions \( \mu_i(x_i) \).
  - Compute: \( Y = f(X_1, \ldots, X_n) \), where \( Y \) is defined by Zadeh's extension principle:
    \[
    \mu(y) = \max_{x_1,\ldots,x_n:f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1),\ldots,\mu_n(x_n)).
    \]

- **Idea:** represent each \( X_i \) by its \( \alpha \)-cuts
  \[
  X_i(\alpha) = \{ x_i : \mu_i(x_i) \geq \alpha \}.
  \]

- **Advantage:** for continuous \( f \), for every \( \alpha \), we have
  \[
  Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)).
  \]

- **Resulting algorithm:** for \( \alpha = 0, 0.1, 0.2, \ldots, 1 \) apply interval computations techniques to compute \( Y(\alpha) \).
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33. Proof of the Result about Chips

- Let us fix the optimal distributions for \( x_2, \ldots, x_n \); then,

\[
\text{Prob}(D \leq y_0) = \sum_{(x_1,\ldots,x_n): D(x_1,\ldots,x_n) \leq y_0} p_1(x_1) \cdot p_2(x_2) \cdot \ldots
\]

- So, \( \text{Prob}(D \leq y_0) = \sum_{i=0}^{N} c_i \cdot q_i \), where \( q_i \overset{\text{def}}{=} p_1(v_i) \).

- Restrictions: \( q_i \geq 0 \), \( \sum_{i=0}^{N} q_i = 1 \), and \( \sum_{i=0}^{N} q_i \cdot v_i = E_1 \).

- Thus, the worst-case distribution for \( x_1 \) is a solution to the following linear programming (LP) problem:

\[
\text{Minimize } \sum_{i=0}^{N} c_i \cdot q_i \text{ under the constraints } \sum_{i=0}^{N} q_i = 1 \text{ and } \sum_{i=0}^{N} q_i \cdot v_i = E_1, \quad q_i \geq 0, \quad i = 0, 1, 2, \ldots, N.
\]
34. Proof of the Result about Chips (cont-d)

- **Minimize:** $\sum_{i=0}^{N} c_i \cdot q_i$ under the constraints $\sum_{i=0}^{N} q_i = 1$ and $\sum_{i=0}^{N} q_i \cdot v_i = E_1$, $q_i \geq 0$, $i = 0, 1, 2, \ldots, N$.

- **Known:** in LP with $N + 1$ unknowns $q_0, q_1, \ldots, q_N$, $\geq N + 1$ constraints are equalities.

- **In our case:** we have 2 equalities, so at least $N - 1$ constraints $q_i \geq 0$ are equalities.

- Hence, no more than 2 values $q_i = p_1(v_i)$ are non-0.

- If corresponding $v$ or $v'$ are in $(x_1, \overline{x_1})$, then for $[v, v'] \subset x_1$ we get the same $y_0$ – in contradiction to non-degeneracy.

- Thus, the worst-case distribution is located at $x_1$ and $\overline{x_1}$.

- The condition that the mean of $x_1$ is $E_1$ leads to the desired formulas for $p_1$ and $\overline{p_1}$. 