From Fuzzification and Intervalization to Anglification: A New 5D Geometric Formalism for Physics and Data Processing

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1. **Data Processing: Geometric Interpretation is Needed**

- *Data* to be *processed*: several real numbers $x_1, \ldots, x_n$.

- *Geometric interpretation*: the sequence $(x_1, \ldots, x_n)$ is an $n$-D vector – an element of an $n$-D space.

- *Resulting visualization*:
  - level sets of Gaussian distribution are ellipsoids;
  - linear relation is a plane, etc.

- *Problem*: we can only use geometric intuition for $\leq 3$ (or 4).

- *Objective*: to have similar geometric techniques for larger $n$.

- *Idea*: look at physics where multi-dimensional geometries are currently used.
2. **Physics: 5D Geometry is Useful**

- General relativity (GRT) explained gravitation by combining space and time into a 4D space.
- *Question:* can other dimensions explain other physics?
- *Success* (Th. Kaluza, O. Klein, 1921): 5D GRT
  - gravitation for \(4 \times 4\) components \(g_{ij}\) of the metric,
  - \(g^{5i}\) satisfy Maxwell’s equations (if \(g_{55} = \text{const}\)).
- *Problem:* no physical explanation of 5-th dimension.
- *Solution* (A. Einstein, P. Bergmann, 1938): 5th dimension forms a tiny circle, so we don’t notice it.
- *This is still relevant:* this idea is standard in particle physics, where
  - space is 10- or 11-dimensional,
  - all dimensions except the first four are tiny.
3. The Physical Model is Unusual, But This Un-Usualness is Appropriate for Data Processing

- **Problem:** the standard multi-D physical model is unusual geometrically:
  - the space is a cylinder,
  - not a plane anymore.

- **Observation:** this feature is, however, interestingly related to data processing:
  - some measured data are angles, and
  - angles do form a circle.

- **Conclusion:** these geometric ideas can be directly applied to data processing.
4. **Geometry Needed**

- **Problem:** Kaluza-Klein theory requires several additional physical formulas w/o geometric meaning.
- **Objective:** we show that these formulas can be geometrically explained.
- First, the assumption $g_{55} = \text{const}$ is artificial.
- Second, since only 4 coordinates have a physical sense, the terms $g_{5i} \cdot \Delta x^5 \cdot \Delta x^i$ in the distance
  \[ \Delta s^2 = \sum_{i=1}^{5} \sum_{j=1}^{5} g_{ij} \cdot \Delta x_i \cdot \Delta x_j \]
  are not physical.
- Third, the observed values of physical fields do not depend on $x^5$ (cylindricity).
- Rumer interpreted $x^5$ as action $S = \int L \, dx \, dt$.
- Fourth, action transformations $S \to S + f(x^i)$ should be geometrically meaningful.
5. **Natural Idea and Its Problems**

- **Main difference:**
  - in Einstein-Bergmann's 5D model we have a cylinder $K = \mathbb{R}^4 \times S^1$ ($K$ for Kaluza)
  - in a standard 4D space, we have a linear space.

- **Idea:** modify standard geometry by substituting $K$ instead of $\mathbb{R}^4$ into all definitions.

- **Problem:** we need linear space structure, i.e., addition and multiplication by a scalar.

- We still have addition in $K$.

- However, multiplication is not uniquely defined for angle-valued variables:
  - we can always interpret an angle as a real number modulo the circumference,
  - but then, e.g., $0 \sim 2\pi$ while $0.6 \cdot 0 \not\sim 0.6 \cdot 2\pi$. 

6. What We Suggest

- **We need**: a real-number representation of an angle variable.
- **Natural idea**: an angle is not as a *single* real number.
- It is a set \( \{ \alpha + n \cdot 2\pi \} \) of all possible real numbers that correspond to the given angle.
- **Similar ideas**: interval and fuzzy arithmetic.
- **Natural definition**: element-wise operations, e.g.,
  \[
  A + B = \{ a + b \mid a \in A, b \in B \}.
  \]
- **Other ideas**:
  - tensors are linear mappings that preserve the structure of such sets;
  - a tensor field is differentiable if its derivatives are also consistent with this structure.
7. Resulting Formalism: Idea

- In mathematical terms, the resulting formalism is equivalent to the following:
- We start with the space $K$ which is not a vector space (only an Abelian group).
- We reformulate standard definitions of vector and tensor algebra and tensor analysis and apply them to $K$:
  - $K$-vectors are defined as elements of $K$;
  - $K$-covectors as elements of the dual group,
  - etc.
- All physically motivated conditions turn out to be natural consequences of this formalism.
8. **$K$-Vectors**

- In the traditional 4-D space-time $R^4$, we can define a *vector* as simply an element of $R^4$.
- In our case, instead of 4-D space-time $R^4$, we have a 5-D space-time $K \overset{\text{def}}{=} R^4 \times S^1$.
- $S^1$ is a circle of a small circumference $h > 0$ – i.e., equivalently, a real line in which two numbers differing by a multiple of $h$ describe the same point: $(x^1, \ldots, x^4, x^5) \sim (x^1, \ldots, x^4, x^5 + k \cdot h)$.
- Thus, it is natural to define $K$-*vectors* as simply elements of $K$.
- On $R^4$, there are two operations: $a + b$ and $\lambda$: $a \rightarrow \lambda \cdot a$. Thus, $R^4$ is a *linear space*.
- On $K$ we only have addition, so $K$ is only an *Abelian group*. 
9. Towards $K$-Covectors

- Vectors describe location $x$, covectors $p$ describe momentum.
- Heisenberg’s principle $\Delta x \cdot \Delta p \geq \hbar$: if we know the momentum, then we have no information about the location.
- Corollary: a state with a definite momentum $p$ does not change under shift $x \rightarrow x + t$.
- In QM, a state is a wave function $\psi(x)$.
- Only probabilities $|\psi|^2$ are observables, so $\psi$ and $\exp(i \cdot \alpha) \cdot \psi$ is the same state.
- Conclusion: $\psi(x + t) = \varphi(t) \cdot \psi(x)$ for $|\varphi(t)| = 1$.
- $\psi(t) = \varphi(t) \cdot \psi(0)$, so we must find $\varphi(t)$.
- $\varphi(t + s) = \varphi(t) \cdot \varphi(s) - \text{homomorphism } R \rightarrow S^1$. 
10.  \( K \)-Covectors

- **Definition:** a \( K \)-covector is a continuous homomorphism from \( K \) to \( S^1 \).

- By a sum of two covectors we mean the product of the corresponding homomorphisms.

- The set of all \( K \)-covectors is thus a dual group \( K^* = R^4 \times Z \) to \( K \).

- It is known that elements of \( K^* \) have the form

  \[
  \exp(i \cdot p \cdot x),
  \]

  where \( p = (p_1, \ldots, p_4, p_5) \) and \( p_5 \) is an multiple of \( 1/h \).

- \( K \)-vectors are vectors \( x = (x_1, \ldots, x_5) \) of \( R^5 \) modulo \( x \sim x' \) if \( x_5 - x'_5 = k \cdot h \) for some integer \( k \).

- \( K \)-covectors are linear mappings that are consistent with the above structure: \( x \sim x' \) implies \( p \cdot x \sim p \cdot x' \).
11. **K-Tensors: Definitions**

- **Tensors** are multi-linear mappings:
  \[
  x^{i_1}, \ldots, y^{i_p}, z_{j_1}, \ldots, u_{j_q} \rightarrow \\
  \sum_{i_1, \ldots, i_p, j_1, \ldots, j_q} t^{j_1 \ldots j_q}_{i_1 \ldots i_p} \cdot x^{i_1} \cdot \ldots \cdot y^{i_p} \cdot z_{j_1} \cdot \ldots \cdot u_{j_q}.
  \]

- A **K-tensor** is a multi-linear mapping that is consistent with the equivalence sets structure, i.e., for which
  - if \( x \sim x', \ldots, y \sim y' \),
  - then \( t(x, \ldots, y, z, \ldots, u) \sim t(x', \ldots, y', z, \ldots, u) \).

- Two multi-linear mappings \( t \) and \( t' \) describe the same **K-tensor** if
  \[
  t(x, \ldots, y, z, \ldots, u) \sim t'(x, \ldots, y, z, \ldots, u)
  \]
  for all \( x, \ldots, y, z, \ldots, u \).
12. **$K$-Tensors: Main Result**

- In a $K$-tensor, of all the components in which one of the lower indices is 5,
  - only a component $t^5...5$ can be non-zero, and
  - this component can only take values $2 \cdot \pi \cdot h^{q-1} \cdot k$ for some integer $k$.

- Two sets of components $t...5...$ and $s...5...$ define the same $K$-tensor if and only if:
  - all their components coincide,
  - with a possible exception of components $t^5...5$ and $s^5...5$ which may differ by $2 \cdot \pi \cdot h^q \cdot k$ for an integer $k$. 
13. Explaining the Condition $g_{55} = \text{const}$ and the Fact that Metric Does Not Depend on $x^5$

- For $g_{ij}$, the above result implies that $g_{55} = g_{5i} = 0$.
- Thus, the above geometric formalism explains the first two physical assumptions that we wanted to explain:
  - that $g_{55} = 0$, and
  - that the distance $\Delta s^2 = \sum_{i=1}^{5} \sum_{j=1}^{5} g_{ij} \cdot \Delta x_i \cdot \Delta x_j$ between the two points $x$ and $x + \Delta x$ only depends on their first 4 coordinates.

- **Definition:** a $K$-tensor field $t^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ is differentiable if its gradient $\partial t^{j_1 \ldots j_q}_{i_1 \ldots i_p} / \partial x^m$ is also a $K$-tensor field.

- **Theorem:** The $K$-tensor field is differentiable if and only if:
  - all its components $t^{\ldots \ldots}_{\ldots \ldots}$ do not depend on $x^5$,
  - with the possible exception of the component $t^{5 \ldots 5}$ which may have the form $2 \cdot \pi \cdot h^{q-1} \cdot x^5 + f(x_1, \ldots, x_4)$.

- **Conclusion:** for all the components $t$ (except for angular-valued ones), we have the cylindricity condition $\partial t^{\ldots \ldots} / \partial x^5 = 0$.

- Thus, the cylindricity conditions is also explained by the geometric model.
15. Coordinate Transformations

- **Observation:** in the traditional geometry, linear coordinates transformations are continuous automorphisms of the additive group $K_0 = \mathbb{R}^4$.

- **Definition:** a $K$-linear transformation is a continuous automorphism of $K$.

- **Description:**

$$x^5 \rightarrow \pm x^5 + \sum_{i=1}^{4} A_i \cdot x^i, \quad x^i \rightarrow \sum_{j=1}^{4} b^i_j x^j, \quad (i \leq 4).$$

- A smooth transformation $s : K \rightarrow K$ is *admissible* iff all tangent transformations are $K$-linear.

- **Description:** every admissible transformation has the form $x^5 \rightarrow \pm x^5 + f(x^1, \ldots, x^4), \quad x^i \rightarrow f^i(x^1, \ldots, x^4)$.

- **Conclusion:** we have exactly 4D transformations and Rumer’s transformations $x^5 \rightarrow x^5 + f(x^1, \ldots, x^4)$. 
16. Potential Applications to Data Processing

For example, a natural analog of Gaussian distribution is \( \exp(-\sum a_{ij} x^i x^j) \) for a \( K \)-tensor \( a_{ij} \).
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