

Towards a General Description of Interval Multiplications: Algebraic Analysis and Its Relation to t-Norms

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1. Outline

- *Interval computations*: are practically important
 - for processing data known with interval uncertainty, and
 - for processing fuzzy data.
- *In general*: interval computations is NP-hard.
- *As a result*: several methods exist:
 - faster less accurate methods, and
 - slower, more accurate methods.
- To select the best method, it is desirable to know all possible methods.
- *What we do*: describe all possible operations for interval multiplication.
- This description is based on the same ideas as a known description of *t-norms in fuzzy logic*.

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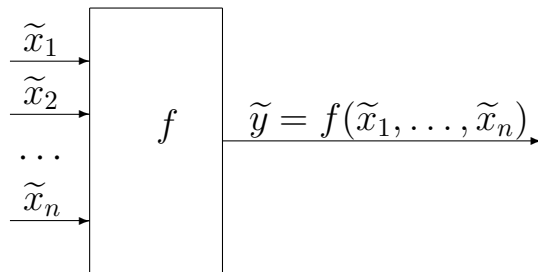


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2. Need for Data Processing

- *Situation*: some quantities y are difficult to measure.
- *Examples*: distance to a star, amount of oil in a well.
- *Solution*:
 - measure related easier-to-measure quantities x_1, \dots, x_n which are related to y by a known algorithm $y = f(x_1, \dots, x_n)$;
 - use measurement results $\tilde{x}_1, \dots, \tilde{x}_n$ to estimate y as $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$:



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3. Need for Interval Computations

- Measurements are never absolutely accurate:

$$\Delta x_i \stackrel{\text{def}}{=} \tilde{x}_i - x_i \neq 0.$$

- *Traditional approach*: assumes that we know the probabilities of Δx_i (from comparing with standard measuring instruments).
- *Situation*: often, we only know upper bounds Δ_i on Δx_i : $|\Delta x_i| \leq \Delta_i$.
- *Examples*: cutting-edge measurements, manufacturing.
- *Enter intervals*: we only know that

$$x_i \in \mathbf{x}_i \stackrel{\text{def}}{=} [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i].$$

- *Problem*: find the range of possible values of

$$y = f(x_1, \dots, x_n)$$

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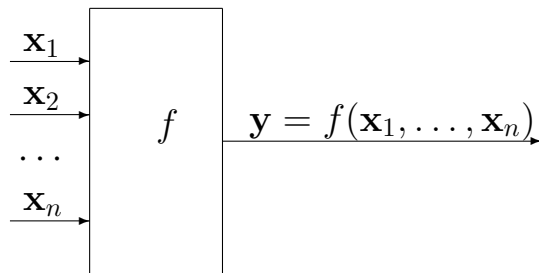
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4. Main Problem of Interval Computations: Reminder

- *Given:* n intervals $\mathbf{x}_1 = [\underline{x}_1, \bar{x}_1], \dots, \mathbf{x}_n = [\underline{x}_n, \bar{x}_n]$.
- *Given:* an (algorithmic) function $f(x_1, \dots, x_n)$.
- *Estimate:* the range

$$[\underline{y}, \bar{y}] = f(\mathbf{x}_1, \dots, \mathbf{x}_n) =$$

$$\{f(x_1, \dots, x_n) : x_1 \in [\underline{x}_1, \bar{x}_1], \dots, x_n \in [\underline{x}_n, \bar{x}_n]\}.$$



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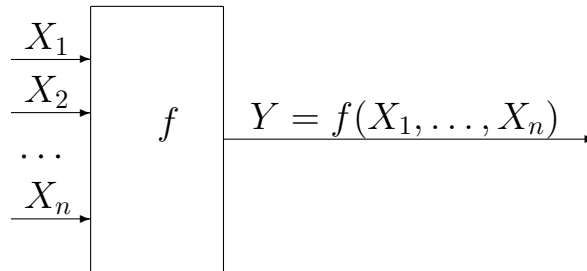


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5. From Interval to Fuzzy Data Processing

- *Given:* n fuzzy numbers X_1, \dots, X_n .
- *Given:* an (algorithmic) function $f(x_1, \dots, x_n)$.
- *Compute:* the fuzzy number $Y = f(X_1, \dots, X_n)$



- *Solution:* consider α -cuts $X_1(\alpha), \dots, X_n(\alpha)$.
- *Known result:* $Y(\alpha) = f(X_1(\alpha), \dots, X_n(\alpha))$.
- *Conclusion:* interval computations are useful in fuzzy data processing.

6. Formulation of the Problem

- *Known fact:* in general, computing the range $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is computationally difficult (NP-hard) even for quadratic f .
- *Practical consequence:* since we cannot compute the exact range, we need *approximate* computations.
- *Trade-off:* there exist different techniques
 - some faster but less accurate;
 - some more accurate but slower.
- *Objective:* find the best technique for each situation.
- *Auxiliary problem:* describe all such techniques.
- *Our result:* we describe all such techniques for the case of multiplication

$$f(x_1, x_2) = x_1 \cdot x_2.$$

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7. Interval Multiplication: Known Operations

- *Problem:* given \mathbf{x}_1 and \mathbf{x}_2 , find

$$\mathbf{Y} \supseteq \mathbf{x}_1 \cdot \mathbf{x}_2 \stackrel{\text{def}}{=} \{x_1 \cdot x_2 : x_1 \in \mathbf{x}_1, x_2 \in \mathbf{x}_2\}.$$

- *Exact formula:* for $\mathbf{x}_1 = [\underline{x}_1, \bar{x}_1]$ and $\mathbf{x}_2 = [\underline{x}_2, \bar{x}_2]$, we take $\mathbf{Y} = [\underline{y}, \bar{y}] = \mathbf{x}_1 \cdot \mathbf{x}_2$, where

$$\underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \bar{x}_2, \bar{x}_1 \cdot \underline{x}_2, \bar{x}_1 \cdot \bar{x}_2),$$

$$\bar{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \bar{x}_2, \bar{x}_1 \cdot \underline{x}_2, \bar{x}_1 \cdot \bar{x}_2).$$

- *Rump's arithmetic:* for $\mathbf{x}_1 = [\tilde{x}_1 - \Delta_1, \tilde{x}_1 + \Delta_1]$ and $\mathbf{x}_2 = [\tilde{x}_2 - \Delta_2, \tilde{x}_2 + \Delta_2]$, we take $\mathbf{Y} = [\tilde{y} - \Delta, \tilde{y} + \Delta]$, where

$$\tilde{y} \stackrel{\text{def}}{=} \tilde{x}_1 \cdot \tilde{x}_2$$

and

$$\Delta \stackrel{\text{def}}{=} \tilde{x}_1 \cdot \Delta_2 + \tilde{x}_2 \cdot \Delta_1 + \Delta_1 \cdot \Delta_2.$$

- Other interval multiplication operations have also been proposed and successfully used.

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8. What Is an Interval Multiplication Operation? Natural Requirements

- We need a function $\mathbf{x}_1 \cdot \mathbf{x}_2$ that maps intervals $\mathbf{x}_1 = [\underline{x}_1, \bar{x}_1]$ and $\mathbf{x}_2 = [\underline{x}_2, \bar{x}_2]$ into a new interval $\mathbf{y} = [\underline{y}, \bar{y}]$.
- *Commutativity*:
 - *Fact*: multiplication is commutative.
 - *Related requirement on interval multiplication operation*: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ for all \mathbf{a} and \mathbf{b} .
- *Associativity*: $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ for all \mathbf{a} , \mathbf{b} , and \mathbf{c} .
- *\subseteq -monotonicity*: if $\mathbf{a} \subseteq \mathbf{a}'$ and $\mathbf{b} \subseteq \mathbf{b}'$ then $\mathbf{a} \cdot \mathbf{b} \subseteq \mathbf{a}' \cdot \mathbf{b}'$.
- *Scalar multiplication*: the operation should produce the exact range when a is a number (a degenerate interval):
 - if $a \geq 0$, then $[a, a] \cdot [\underline{b}, \bar{b}] = [a \cdot \underline{b}, a \cdot \bar{b}]$;
 - if $a < 0$, then $[a, a] \cdot [\underline{b}, \bar{b}] = [a \cdot \bar{b}, a \cdot \underline{b}]$.

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9. A 1-D Analogue of This Problem and T-Norms

- *A similar 1-D problem:* classify all monotonic associative operations $a * b$ on real numbers.
- *Known solution:* classification of t-norms (&-operations).
- *Reminder:* a t-norm is a commutative associative monotonic operation $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for which $t(1, 1) = 1$ and $t(0, 0) = t(0, 1) = 0$.
- *First result:* some t-norms have the form

$$a * b = F^{-1}(F(a) + F(b))$$

for some $F(x)$.

- *Second result:* all t-norms can be represented as limits of such t-norms.
- *Example:* $\min(a, b) = \lim_{t \rightarrow \infty} (a^{-t} + b^{-t})^{-1/t}$.
- *Comment:* the above formula can be reformulated as $F(a * b) = F(a) + F(b)$.

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10. Classification Problem for Interval Multiplication Operations: First Simplification

- *Starting point:* we need two functions of 4 variables: $\underline{y} = \underline{y}(\underline{x}_1, \bar{x}_1, \underline{x}_2, \bar{x}_2)$ and $\bar{y} = \bar{y}(\underline{x}_1, \bar{x}_1, \underline{x}_2, \bar{x}_2)$.

- *Idea:* every interval $\mathbf{x}_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ with $\tilde{x}_i > 0$ can be represented as $\tilde{x}_i \cdot I(\delta_i)$, where $\delta_i \stackrel{\text{def}}{=} \Delta_i / \tilde{x}_i$ and

$$I(\delta_i) \stackrel{\text{def}}{=} [1 - \delta_i, 1 + \delta_i].$$

- *Consequence:* due to commutativity and associativity, we have $\mathbf{x}_1 \cdot \mathbf{x}_2 = (\tilde{x}_1 \cdot \tilde{x}_2) \cdot [I(\delta_1) \cdot I(\delta_2)]$.
- *Reminder:* the product of an interval $I(\delta_1) \cdot I(\delta_2)$ and a real number $\tilde{x}_1 \cdot \tilde{x}_2$ is equal to the actual range.
- *Conclusion:* it is sufficient to describe the product $[\underline{y}, \bar{y}] = I(\delta_1) \cdot I(\delta_2)$ of the intervals $I(\delta_1)$ and $I(\delta_2)$.
- *Simplification:* we need two functions of 2 variables: $\underline{y} = \underline{y}(\delta_1, \delta_2)$ and $\bar{y} = \bar{y}(\delta_1, \delta_2)$.

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11. Examples

- *General case:* we are interested in

$$I(\delta_1) \cdot I(\delta_2) = [1 - \delta_1, 1 + \delta_1] \cdot [1 - \delta_2, 1 + \delta_2].$$

- *Case of standard interval multiplication:* when $\delta_i < 1$, we have

$$\begin{aligned} I(\delta_1) \cdot I(\delta_2) &= [1 - \delta_1, 1 + \delta_1] \cdot [1 - \delta_2, 1 + \delta_2] = \\ &= [(1 - \delta_1) \cdot (1 - \delta_2), (1 + \delta_1) \cdot (1 + \delta_2)]. \end{aligned}$$

- *Case of Rump's arithmetic:*

$$\begin{aligned} I(\delta_1) \cdot I(\delta_2) &= [1 - \delta_1, 1 + \delta_1] \cdot [1 - \delta_2, 1 + \delta_2] = \\ &= [1 - (\delta_1 + \delta_2 + \delta_1 \cdot \delta_2), 1 + (\delta_1 + \delta_2 + \delta_1 \cdot \delta_2)], \end{aligned}$$

i.e., $I(\delta_1) \cdot I(\delta_2) = I(\delta)$, where $\delta = \delta_1 + \delta_2 + \delta_1 \cdot \delta_2$.

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12. Reduction to One Function of Two Variables and One Function of One Variable

- *Reminder:* each interval \mathbf{x}_i is a product of its midpoint \tilde{x}_i and an interval $I(\delta_i)$ centered at 1.
- *Example:* $I(\delta_1) \cdot I(\delta_2) = m(\delta_1, \delta_2) \cdot I(w(\delta_1, \delta_2))$.
- *Comment:* relative half-width $\delta_i = \Delta_i/\tilde{x}_i$ does not change if multiply an interval by a number.
- *Conclusion:* the relative half-width δ of $\mathbf{y} = \mathbf{x}_1 \cdot \mathbf{x}_2 = \text{const} \cdot I(\delta_1) \cdot I(\delta_2)$ is equal to $\delta = w(\delta_1, \delta_2)$.
- *Conclusion:* relative width of the product of 3 intervals is $w(w(\delta_1, \delta_2), \delta_3)$.
- *Associativity:* implies that w is associative, so $w(a, b) = F^{-1}(F(a) + F(b))$ (or a limit case), and
$$I(\delta_1) \cdot I(\delta_2) = m(\delta_1, \delta_2) \cdot I(F^{-1}(F(\delta_1) + F(\delta_2))).$$
- *Result:* we need $m(\delta_1, \delta_2)$ and $F(a)$.

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13. Examples

- *Case of standard interval multiplication:*

$$m(\delta_1, \delta_2) = 1 + \delta_1 \cdot \delta_2; \quad w(\delta_1, \delta_2) = \frac{\delta_1 + \delta_2}{1 + \delta_1 \cdot \delta_2}.$$

- *Comment:* same formula as adding velocities in Special Relativity (in units in which $c = 1$).
- *In terms of F :* $w(\delta_1, \delta_2) = F^{-1}(F(\delta_1) + F(\delta_2))$, where $F(a)$ is an inverse function to \tanh , i.e.,

$$F(a) = \tanh^{-1}(a) = \frac{1}{2} \cdot \ln \left(\frac{1+a}{1-a} \right).$$

- *Case of Rump's arithmetic:*

$$m(\delta_1, \delta_2) = 1; \quad w(\delta_1, \delta_2) = \delta_1 + \delta_2 + \delta_1 \cdot \delta_2.$$

- *In terms of F :* here, $1 + \delta = (1 + \delta_1) \cdot (1 + \delta_2)$, hence $\ln(1 + \delta) = \ln(1 + \delta_1) + \ln(1 + \delta_2)$, and so $w(\delta_1, \delta_2) = F^{-1}(F(\delta_1) + F(\delta_2))$ for $F(a) = \ln(1 + a)$.

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14. Reduction to Two Functions of One Variable

- *Reminder:* $I(\delta_1) \cdot I(\delta_2) = m(\delta_1, \delta_2) \cdot I(\delta_1 * \delta_2)$, where we denoted $\delta_1 * \delta_2 \stackrel{\text{def}}{=} F^{-1}(F(\delta_1) + F(\delta_2))$.

- *Associativity* implies

$$m(\delta_1, \delta_2) \cdot m(\delta_1 * \delta_2, \delta_3) = m(\delta_1, \delta_2 * \delta_3) \cdot m(\delta_2, \delta_3).$$

- *Simplification:* in terms of $a_i \stackrel{\text{def}}{=} F(\delta_i)$, the operation $\delta = \delta_1 * \delta_2$ is $a = a_1 + a_2$, so for $\ell(a_1, a_2) = \log(M(a_1, a_2))$, where $M(a_1, a_2) \stackrel{\text{def}}{=} m(F^{-1}(\delta_1), F^{-1}(\delta_2))$, we get

$$\ell(a_1, a_2) + \ell(a_1 + a_2, a_3) = \ell(a_1, a_2 + a_3) + \ell(a_2, a_3).$$

- $\ell(a_1, a_2)$ is a 2-cocycle, hence it is co-boundary: $\ell(a_1, a_2) = g(a_1) + g(a_2) - g(a_1 + a_2)$ for some $g(a)$.

- *Conclusion:* $M(a_1, a_2) = \exp(\ell(a_1, a_2)) = \frac{e^{g(a_1+a_2)}}{e^{g(a_1)} \cdot e^{g(a_2)}}$

$$\text{and } m(\delta_1, \delta_2) = \frac{\varphi(\delta_1 * \delta_2)}{\varphi(\delta_1) \cdot \varphi(\delta_2)}, \text{ where } \varphi(\delta) \stackrel{\text{def}}{=} e^{g(F(\delta_i))}.$$

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15. Resulting General Description of Interval Multiplication Operations

- *Reminder:* $I(\delta_1) \cdot I(\delta_2) = m(\delta_1, \delta_2) \cdot I(\delta_1 * \delta_2)$, and we have an expression for $m(\delta_1, \delta_2)$ in terms of $\varphi(\delta)$.

- *Resulting formulas:*

$$I(\delta_1) \cdot I(\delta_2) = \frac{\varphi(\delta_1 * \delta_2)}{\varphi(\delta_1) \cdot \varphi(\delta_2)} \cdot I(\delta_1 * \delta_2),$$

where $\delta_1 * \delta_2 = F^{-1}(F(\delta_1) + F(\delta_2))$.

- *Comment:* vice versa, for arbitrary monotonic functions $F(a)$ and $\varphi(\delta)$, the above formula defines an interval multiplication operation.
- *Observation:*
 - usually, interval computations help in fuzzy computing;
 - here, vice versa, a known fact from fuzzy theory helps interval computations.

16. Application to Stochastic Arithmetic

- *Case of stochastic arithmetic:* the mean \tilde{x} and a standard deviation σ form a confidence interval

$$[\tilde{x} - k \cdot \sigma, \tilde{x} + k \cdot \sigma].$$

- *Problem:* given \tilde{x}_i and σ_i of two independent variables x_1 and x_2 , to find the mean \tilde{y} and the standard deviation σ of their product.
- *Formulas:* $\tilde{y} = \tilde{x}_1 \cdot \tilde{x}_2$ and $\sigma = \sqrt{\tilde{x}_1^2 \cdot \sigma_2^2 + \tilde{x}_2^2 \cdot \sigma_1^2 + \sigma_1^2 \cdot \sigma_2^2}$.
- *This operation is a particular case of our general description:* $m(\delta_1, \delta_2) = 1$, $w(\delta_1, \delta_2) = \sqrt{1 + \delta_1^2 + \delta_2^2 + \delta_1^2 \cdot \delta_2^2}$.
- *Discussion:* this expression is equivalent to $(1 + \delta^2) = (1 + \delta_1^2) \cdot (1 + \delta_2^2)$, thus $\ln(1 + \delta^2) = \ln(1 + \delta_1^2) + \ln(1 + \delta_2^2)$.
- *Result:* $w(\delta_1, \delta_2) = F^{-1}(F(\delta_1) + F(\delta_2))$ for

$$F(\delta) = \ln(1 + \delta^2).$$

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