Square Root of “Not”: A Major Difference Between Fuzzy and Quantum Logics

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1. Quantum Logic and Fuzzy Logic

- Both quantum logic and fuzzy logic describe uncertainty:
  - quantum logic describes uncertainties of the real world;
  - fuzzy logic described the uncertainty of our reasoning.
- Due to this common origin, there is a lot of similarity between the two logics.
- These similarities have been emphasized in several papers on fuzzy logic (Kosko et al.).
- *What we plan to do:* emphasize difference.
- *Specifically:* only in quantum logic there is a "square root of not" operation $s(a)$:

$$s(s(a)) = \neg a \text{ for all } a.$$

2. There Is No Square Root of Not in Classical Logic

- In classical logic, we have 2 truth values: "true" (1) and "false" (0).
- In classical logic, a unary operation $s(a)$ can be described by listing its values $s(0)$ and $s(1)$.
- There are two possible values of $s(0)$ and two possible values of $s(1)$.
- So overall, we have $2 \times 2 = 4$ possible unary operations:
  - when $s(0) = s(1) = 0$, then we get a constant function whose value is "false";
  - when $s(0) = s(1) = 1$, then we get a constant function whose value is "true";
  - when $s(0) = 0$ and $s(1) = 1$, we get the identity function;
  - finally, when $s(0) = 1$ and $s(1) = 0$, we get the negation.

3. There Is No Square Root of Not in Classical Logic (cont-d)

- *Reminder:* there are 4 unary functions $s(a)$: constant false, constant true, identity, and negation.
- In all four cases, the composition $s(s(a))$ is different from the negation:
  - for the "constant false" function s, we have $s(s(a)) = s(a)$, i.e., $s(s(a))$ is also the constant false function;
  - for the "constant true" function s, also $s(s(a)) = s(a)$, i.e., $s(s(a))$ is also the constant true function;
  - for the identity function s, we have $s(s(a)) = s(a)$, i.e., the composition of s and s is also the identity;
  - finally, for the negation s, the composition of s and s is the identity function.

4. Quantum Mechanics

- To adequately describe microparticles, we need quantum mechanics.
- One of the main features of quantum mechanics is the possibility of *superpositions*.
- A superposition $s \stackrel{\text{def}}{=} c_1 \cdot |s_1\rangle + \ldots + c_n \cdot |s_n\rangle$ "combines" states $|s_1\rangle, \ldots, |s_n\rangle$.
- Measuring $|s_i\rangle$ in s leads to s_i with probability $|c_i|^2$.
- The total probability is 1, hence $|c_1|^2 + \ldots + |c_n|^2 = 1$.
- If we multiply all c_i by the same constant $e^{\mathrm{i}\cdot\alpha}$ (with real α), we get the same outcome probabilities.
- In quantum mechanics, states s and $e^{\mathrm{i}\cdot\alpha} \cdot s$ are therefore considered the same physical state.

5. Quantum Logic

- Quantum Logic is an application of the general idea of quantum mechanics to logic.
- In the classical logic, there are two possible states: $|0\rangle$ and $|1\rangle$, with $\neg(|0\rangle) = |1\rangle$ and $\neg(|1\rangle) = |0\rangle$.
- In quantum logic, can also have superpositions

$$c_0 \cdot |0\rangle + c_1 \cdot |1\rangle \text{ when } |c_0|^2 + |c_1|^2 = 1.$$

- These superpositions are the "truth values" of quantum logic.
- In general, in quantum mechanics, all operations are linear in terms of superpositions.
- By using this linearity, we can describe the negation of an arbitrary quantum state:

$$\neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = c_0 \cdot |1\rangle + c_1 \cdot |0\rangle.$$

6. Alternative Quantum Negation

- *Alternative description:*

$$\neg(|0\rangle) = -|1\rangle; \quad \neg(|1\rangle) = |0\rangle.$$

- *Idea:* $-|1\rangle$ and $|1\rangle$ is the same physical state.
- By using this linearity, we can describe the negation of an arbitrary quantum state:

$$\neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = -c_0 \cdot |1\rangle + c_1 \cdot |0\rangle.$$

- Here,

$$\neg\neg(|0\rangle) = \neg(-|1\rangle) = -|0\rangle;$$
$$\neg\neg(|1\rangle) = \neg(|0\rangle) = -|1\rangle.$$

- Due to linearity, we have

$$\neg\neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = -(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle.$$

- In other words, $\neg\neg(s) = -s$, i.e., $\neg\neg(s)$ and s is the same physical state.

7. Square Root of Not: Case of Alternative Definition

- *Definition: reminder:* $\neg(|0\rangle) = -|1\rangle$ and $\neg(|1\rangle) = |0\rangle$.
- *Geometric interpretation:* negation is rotation by 90 degrees.
- *Natural square root $s(a)$:* rotation by 45 degrees.
- *Resulting formulas* for $|0\rangle$ and $|1\rangle$:

$$s(|0\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle; \quad s(|1\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle.$$

- *Resulting formulas* for the general case:

$$s(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) =$$

$$c_0 \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle \right) + c_1 \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle \right).$$

8. Square Root of Not: Case of Original Definition

- Let us show that in quantum mechanics, there exists an operation s for which $s(s(a)) = \neg(a)$.
- Due to linearity, it is sufficient to define this operation for the basic states $|0\rangle$ and $|1\rangle$:

$$s(|0\rangle) = \frac{1+\mathrm{i}}{\sqrt{2}}\cdot|0\rangle + \frac{1-\mathrm{i}}{\sqrt{2}}\cdot|1\rangle;\ \ s(|1\rangle) = \frac{1-\mathrm{i}}{\sqrt{2}}\cdot|0\rangle + \frac{1+\mathrm{i}}{\sqrt{2}}\cdot|1\rangle.$$

- For $|1\rangle$, we get $s(s(|1\rangle) = s\left(\frac{1-\mathrm{i}}{\sqrt{2}}\cdot|0\rangle + \frac{1+\mathrm{i}}{\sqrt{2}}\cdot|1\rangle\right)$.
- Due to linearity, $s(s(|1\rangle) = \frac{1-\mathrm{i}}{\sqrt{2}}\cdot s(|0\rangle) + \frac{1+\mathrm{i}}{\sqrt{2}}\cdot s(|1\rangle)$.
- Subst. $s(|0\rangle)$ and $s(|1\rangle)$, we get $s(s(|0\rangle)) = |1\rangle = \neg(|0\rangle)$.
- Similarly, we get $s(s(|1\rangle)) = |0\rangle = \neg(|1\rangle)$.
- By linearity, we get $s(s(a)) = \neg(a)$ for all a.

9. Square Root of Not Is An Important Part of Quantum Algorithms

- *Fact:* square root of not is an important part of quantum algorithms.
- *Search* in an unsorted list of size N:
  - without using quantum effects, we need – in the worst case – at least N computational steps;
  - Grover's quantum algorithm can find this element much faster – in $O(\sqrt{N})$ time.
- *Factoring large integers:*
  - without using quantum effects, we need exponential time;
  - Shor's quantum algorithm only requires polynomial time.

10. Why Factoring Large Integers Is Important

- Most security features of online communications and e-commerce use RSA encryption algorithm.
- This algorithm was named after its authors: R. Rivest, A. Shamir, and L. Adleman.
- To decrypt RSA-encrypted messages, one needs to factor large integers.
- At present, this factorization requires exponential time.
- Thus, for 200-digit numbers, we need billions of years to decrypt RSA-encrypted messages.
- So, at present, the RSA algorithm provides safe communication.
- However, quantum computers will leads to breaking most exiting encryption codes.

11. Fuzzy Logic

- In fuzzy logic, in addition to the classical values 0 and 1, we also allow intermediate truth values.
- In fuzzy logic, these intermediate truth values are arbitrary real numbers from the interval $[0, 1]$.
- Usually, in fuzzy logic, negation is defined as

$$\neg(a) = 1 - a.$$

- *Comment:*
  - In principle, there exist other negation operations.
  - However, it is known that they can be reduced to this standard negation by a re-scaling of $[0, 1]$.

12. There Is No Continuous Square Root of Not in Fuzzy Logic: A Statement

- In fuzzy logic, usually, we only consider logical operations which are continuous functions of their inputs.
- *Reason:*
  - the degrees of uncertainty are only approximately knows, and
  - similar values of the input degrees should lead to similar values of the result of the logical operation.
- *Conclusion:* we restrict ourselves to continuous operations $s : [0, 1] \to [0, 1]$.
- *Main result:* in fuzzy logic, there is no (continuous) square root of negation.

13. Proof

- *Idea:* proof by contradiction.
- Assume that $s(s(a)) = 1 - a$ for a continuous $s : [0, 1] \to [0, 1]$.
- *Lemma:* If $a \neq b$, then $s(a) \neq s(b)$.
- *Proof:* if $s(a) = s(b)$, then $s(s(a)) = s(s(b))$, hence $1 - a = 1 - b$ and $a = b$, but we assumed $a \neq b$.
- *Conclusion:* s is a 1-1 function.
- *Known:* every 1-1 continuous function is strictly monotonic.
- *Conclusion:* $s\uparrow$ or $s\downarrow$.
- *Case of* $s\uparrow$*:* $a < b$ implies $s(a) < s(b)$ and thus, $s(s(a)) < s(s(b))$, but $1 - a > 1 - b$.
- *Case of* $s\downarrow$*:* a similar contradiction.

14. Comment: Discontinuous Square Roots of "Not" Are Possible in Fuzzy Logic

If we do not require continuity, then a square root of not $s(x)$ is possible in fuzzy logic.

- when $0 \le x < \frac{1}{4}$, we set $s(x) = x + \frac{1}{4}$;
- when $\frac{1}{4} \le x < \frac{1}{2}$, we set $s(x) = \frac{5}{4} - x$;
- when $x = \frac{1}{2}$, we set $s(x) = \frac{1}{2}$;
- when $\frac{1}{2} < x \le \frac{3}{4}$, we set $s(x) = \frac{3}{4} - x$;
- finally, when $\frac{3}{4} < x \le 1$, we set $s(x) = x - \frac{1}{4}$.

By considering all 5 cases, we can check that $s(s(x)) = x$ for all $x \in [0, 1]$.

15. Conclusions and Future Work

- *Main result:* in spite of the seeming similarity between the two logics, they are different.
- They are different in square root of "not" – crucial for speed-up of quantum computing.
- This difference is not unexpected:
  - fuzzy logic is a human way of reasoning about the real-world phenomena;
  - most real-world phenomena are well described by classical physics;
  - so it is not surprising that our way of reasoning is not well-suited for quantum physics.
- *Auxiliary result:* if we add discontinuity, we get $\sqrt{\text{not}}$.
- *Hope:* by combining intuitive ideas of discontinuity and fuzzy, we can understand complex quantum ideas.

16. Acknowledgments

This work was supported in part:

- by NSF grants HRD-0734825, EAR-0225670, and EIA-0080940,
- by Texas Department of Transportation contract No. 0-5453,
- by the Japan Advanced Institute of Science and Technology (JAIST) International Joint Research Grant 2006-08, and
- by the Max Planck Institut für Mathematik.

The authors are thankful to the anonymous referees for valuable suggestions.