Square Root of “Not”: A Major Difference Between Fuzzy and Quantum Logics

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1. Quantum Logic and Fuzzy Logic

- Both quantum logic and fuzzy logic describe uncertainty:
  - quantum logic describes uncertainties of the real world;
  - fuzzy logic described the uncertainty of our reasoning.
- Due to this common origin, there is a lot of similarity between the two logics.
- These similarities have been emphasized in several papers on fuzzy logic (Kosko et al.).
- What we plan to do: emphasize difference.
- Specifically: only in quantum logic there is a “square root of not” operation $s(a)$:
  $$s(s(a)) = \neg a \text{ for all } a.$$
2. There Is No Square Root of Not in Classical Logic

• In classical logic, we have 2 truth values: “true” (1) and “false” (0).

• In classical logic, a unary operation \( s(a) \) can be described by listing its values \( s(0) \) and \( s(1) \).

• There are two possible values of \( s(0) \) and two possible values of \( s(1) \).

• So overall, we have \( 2 \times 2 = 4 \) possible unary operations:
  - when \( s(0) = s(1) = 0 \), then we get a constant function whose value is “false”;
  - when \( s(0) = s(1) = 1 \), then we get a constant function whose value is “true”;
  - when \( s(0) = 0 \) and \( s(1) = 1 \), we get the identity function;
  - finally, when \( s(0) = 1 \) and \( s(1) = 0 \), we get the negation.
3. There Is No Square Root of Not in Classical Logic (cont-d)

• **Reminder**: there are 4 unary functions \(s(a)\): constant false, constant true, identity, and negation.

• In all four cases, the composition \(s(s(a))\) is different from the negation:
  - for the “constant false” function \(s\), we have \(s(s(a)) = s(a)\), i.e., \(s(s(a))\) is also the constant false function;
  - for the “constant true” function \(s\), also \(s(s(a)) = s(a)\), i.e., \(s(s(a))\) is also the constant true function;
  - for the identity function \(s\), we have \(s(s(a)) = s(a)\), i.e., the composition of \(s\) and \(s\) is also the identity;
  - finally, for the negation \(s\), the composition of \(s\) and \(s\) is the identity function.
4. Quantum Mechanics

• To adequately describe microparticles, we need quantum mechanics.

• One of the main features of quantum mechanics is the possibility of superpositions.

• A superposition \( s = c_1 \cdot |s_1\rangle + \ldots + c_n \cdot |s_n\rangle \) “combines” states \( |s_1\rangle, \ldots, |s_n\rangle \).

• Measuring \( |s_i\rangle \) in \( s \) leads to \( s_i \) with probability \( |c_i|^2 \).

• The total probability is 1, hence \( |c_1|^2 + \ldots + |c_n|^2 = 1 \).

• If we multiply all \( c_i \) by the same constant \( e^{i \alpha} \) (with real \( \alpha \)), we get the same outcome probabilities.

• In quantum mechanics, states \( s \) and \( e^{i \alpha} \cdot s \) are therefore considered the same physical state.
5. **Quantum Logic**

- Quantum Logic is an application of the general idea of quantum mechanics to logic.
- In the classical logic, there are two possible states: $|0\rangle$ and $|1\rangle$, with $\neg(|0\rangle) = |1\rangle$ and $\neg(|1\rangle) = |0\rangle$.
- In quantum logic, can also have superpositions
  \[ c_0 \cdot |0\rangle + c_1 \cdot |1\rangle \text{ when } |c_0|^2 + |c_1|^2 = 1. \]
- These superpositions are the “truth values” of quantum logic.
- In general, in quantum mechanics, all operations are linear in terms of superpositions.
- By using this linearity, we can describe the negation of an arbitrary quantum state:
  \[ \neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = c_0 \cdot |1\rangle + c_1 \cdot |0\rangle. \]
6. Alternative Quantum Negation

- **Alternative description:**
  \[ \neg(|0\rangle) = -|1\rangle; \quad \neg(|1\rangle) = |0\rangle. \]

- **Idea:** \(-|1\rangle\) and \(|1\rangle\) is the same physical state.

- By using this linearity, we can describe the negation of an arbitrary quantum state:
  \[ \neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = -c_0 \cdot |1\rangle + c_1 \cdot |0\rangle. \]

- Here,
  \[ \neg\neg(|0\rangle) = \neg(-|1\rangle) = -|0\rangle; \]
  \[ \neg\neg(|1\rangle) = \neg(|0\rangle) = -|1\rangle. \]

- Due to linearity, we have
  \[ \neg\neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = -(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle). \]

- In other words, \(\neg\neg(s) = -s\), i.e., \(\neg\neg(s)\) and \(s\) is the same physical state.
7. Square Root of Not: Case of Alternative Definition

- **Definition:** reminder: \( \neg(|0\rangle) = -|1\rangle \) and \( \neg(|1\rangle) = |0\rangle \).
- **Geometric interpretation:** negation is rotation by 90 degrees.
- **Natural square root** \( s(a) \): rotation by 45 degrees.
- **Resulting formulas** for \( |0\rangle \) and \( |1\rangle \):
  \[
s(|0\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle; \quad s(|1\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle.
\]
- **Resulting formulas** for the general case:
  \[
s(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) =
  c_0 \cdot \left( \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle \right) + c_1 \cdot \left( \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle \right).
\]
8. **Square Root of Not: Case of Original Definition**

- Let us show that in quantum mechanics, there exists an operation $s$ for which $s(s(a)) = \neg(a)$.
- Due to linearity, it is sufficient to define this operation for the basic states $|0\rangle$ and $|1\rangle$:

$$
s(|0\rangle) = \frac{1+i}{\sqrt{2}}|0\rangle + \frac{1-i}{\sqrt{2}}|1\rangle; \quad s(|1\rangle) = \frac{1-i}{\sqrt{2}}|0\rangle + \frac{1+i}{\sqrt{2}}|1\rangle.
$$

- For $|1\rangle$, we get $s(s(|1\rangle)) = s\left(\frac{1-i}{\sqrt{2}}|0\rangle + \frac{1+i}{\sqrt{2}}|1\rangle\right)$.
- Due to linearity, $s(s(|1\rangle)) = \frac{1-i}{\sqrt{2}} \cdot s(|0\rangle) + \frac{1+i}{\sqrt{2}} \cdot s(|1\rangle)$.
- Subst. $s(|0\rangle)$ and $s(|1\rangle)$, we get $s(s(|0\rangle)) = |1\rangle = \neg(|0\rangle)$.
- Similarly, we get $s(s(|1\rangle)) = |0\rangle = \neg(|1\rangle)$.
- By linearity, we get $s(s(a)) = \neg(a)$ for all $a$. 

9. **Square Root of Not Is An Important Part of Quantum Algorithms**

- **Fact:** square root of not is an important part of quantum algorithms.

- **Search** in an unsorted list of size $N$:
  - without using quantum effects, we need – in the worst case – at least $N$ computational steps;
  - Grover’s quantum algorithm can find this element much faster – in $O(\sqrt{N})$ time.

- **Factoring large integers:**
  - without using quantum effects, we need exponential time;
  - Shor’s quantum algorithm only requires polynomial time.
10. Why Factoring Large Integers Is Important

- Most security features of online communications and e-commerce use RSA encryption algorithm.
- This algorithm was named after its authors: R. Rivest, A. Shamir, and L. Adleman.
- To decrypt RSA-encrypted messages, one needs to factor large integers.
- At present, this factorization requires exponential time.
- Thus, for 200-digit numbers, we need billions of years to decrypt RSA-encrypted messages.
- So, at present, the RSA algorithm provides safe communication.
- However, quantum computers will lead to breaking most exiting encryption codes.
11. Fuzzy Logic

- In fuzzy logic, in addition to the classical values 0 and 1, we also allow intermediate truth values.
- In fuzzy logic, these intermediate truth values are arbitrary real numbers from the interval [0, 1].
- Usually, in fuzzy logic, negation is defined as
  \[ \neg(a) = 1 - a. \]
- **Comment:**
  - In principle, there exist other negation operations.
  - However, it is known that they can be reduced to this standard negation by a re-scaling of [0, 1].
12. There Is No Continuous Square Root of Not in Fuzzy Logic: A Statement

- In fuzzy logic, usually, we only consider logical operations which are continuous functions of their inputs.

- **Reason:**
  - the degrees of uncertainty are only approximately known, and
  - similar values of the input degrees should lead to similar values of the result of the logical operation.

- **Conclusion:** we restrict ourselves to continuous operations $s : [0, 1] \rightarrow [0, 1]$.

- **Main result:** in fuzzy logic, there is no (continuous) square root of negation.
13. Proof

- **Idea**: proof by contradiction.
- **Assume that** $s(s(a)) = 1 - a$ for a continuous $s : [0, 1] \rightarrow [0, 1]$.
- **Lemma**: If $a \neq b$, then $s(a) \neq s(b)$.
- **Proof**: if $s(a) = s(b)$, then $s(s(a)) = s(s(b))$, hence $1 - a = 1 - b$ and $a = b$, but we assumed $a \neq b$.
- **Conclusion**: $s$ is a 1-1 function.
- **Known**: every 1-1 continuous function is strictly monotonic.
- **Conclusion**: $s \uparrow$ or $s \downarrow$.
- **Case of $s \uparrow$**: $a < b$ implies $s(a) < s(b)$ and thus, $s(s(a)) < s(s(b))$, but $1 - a > 1 - b$.
- **Case of $s \downarrow$**: a similar contradiction.
14. **Comment: Discontinuous Square Roots of “Not” Are Possible in Fuzzy Logic**

If we do not require continuity, then a square root of not $s(x)$ is possible in fuzzy logic.

- when $0 \leq x < \frac{1}{4}$, we set $s(x) = x + \frac{1}{4}$;
- when $\frac{1}{4} \leq x < \frac{1}{2}$, we set $s(x) = \frac{5}{4} - x$;
- when $x = \frac{1}{2}$, we set $s(x) = \frac{1}{2}$;
- when $\frac{1}{2} < x \leq \frac{3}{4}$, we set $s(x) = \frac{3}{4} - x$;
- finally, when $\frac{3}{4} < x \leq 1$, we set $s(x) = x - \frac{1}{4}$.

By considering all 5 cases, we can check that $s(s(x)) = x$ for all $x \in [0, 1]$. 
15. **Conclusions and Future Work**

- **Main result:** in spite of the seeming similarity between the two logics, they are different.
- They are different in square root of “not” – crucial for speed-up of quantum computing.
- This difference is not unexpected:
  - fuzzy logic is a human way of reasoning about the real-world phenomena;
  - most real-world phenomena are well described by classical physics;
  - so it is not surprising that our way of reasoning is not well-suited for quantum physics.
- **Auxiliary result:** if we add discontinuity, we get $\sqrt{\text{not}}$.
- **Hope:** by combining intuitive ideas of discontinuity and fuzzy, we can understand complex quantum ideas.
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