Towards Fast Algorithms for Processing Type-2 Fuzzy Data: Extending Mendel's Algorithms From Interval-Valued to a More General Case

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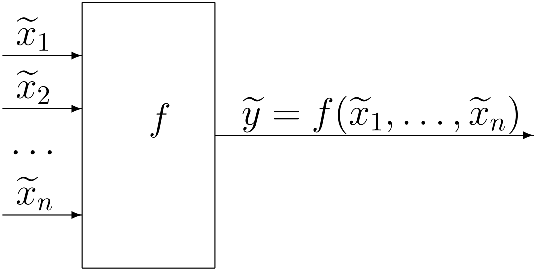
1. Outline

- *Known:* processing (type-1) fuzzy data can be reduced to interval uncertainty:
  - Zadeh's extension principle is equivalent to
  - level-by-level interval computations on α-cuts.
- *More adequate description:* type-2 fuzzy sets.
- *Practical limitation:* transition to type-2 increases computational complexity.
- *Mendel's algorithm:* for interval-valued fuzzy sets, processing can also be reduced to interval computations.
- *In this talk:* we show that Mendel's ideas can be naturally extended to arbitrary type-2 fuzzy numbers.

2. Why Data Processing and Knowledge Processing Are Needed in the First Place

- *Problem:* some quantities y are difficult (or impossible) to measure or estimate directly.
- *Solution:* indirect measurements or estimates



- *Fact:* estimates $\widetilde{x}_i$ are approximate.
- *Question:* how approximation errors $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ affect the resulting error $\Delta y = \widetilde{y} - y$?

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3. From Probabilistic to Interval Uncertainty

- Manufacturers of MI provide us with bounds Δ_i on measurement errors: $|\Delta x_i| \le \Delta_i$.
- Thus, we know that $x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.
- Often, we also know probabilities, but in 2 cases, we don't:
  - cutting-edge measurements;
  - cutting-cost manufacturing.
- In such situations:
  - we know the intervals $[\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ of possible values of x_i, and
  - we want to find the range of possible values of y:

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\}.$$

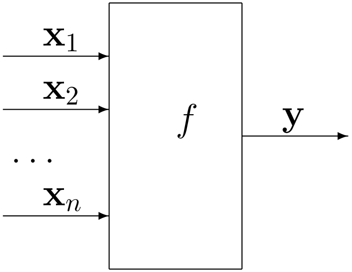
4. Main Problem of Interval Computations

We are given:

- an integer n;
- n intervals $\mathbf{x}_1 = [\underline{x}_1, \overline{x}_1]$, …, $\mathbf{x}_n = [\underline{x}_n, \overline{x}_n]$, and
- an algorithm $f(x_1, \ldots, x_n)$ which transforms n real numbers into a real number $y = f(x_1, \ldots, x_n)$.

We need to compute the endpoints $\underline{y}$ and $\overline{y}$ of the interval

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\}.$$



5. Knowledge Processing and Fuzzy Uncertainty

- In many practical situations, we only have expert estimates for the inputs x_i like "around $\widetilde{x}_i$".
- Such estimates are described by membership functions $m_i(x_i)$ or α-cuts $x_i(\alpha) = \{x : m_i(x) \ge \alpha\}$.
- Usually, each $m_i(x_i)$ is a *fuzzy number*: starts at 0, increases, then decreases to 0.
- For fuzzy numbers, α-cuts are intervals.
- The membership function for $y = f(x_1, \ldots, x_n)$ is defined by Zadeh's extension principle:

  $$m(y) = \sup\{\min(m_1(x_1), \ldots, m_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

- For fuzzy numbers, computations reduce to layer-by-layer interval computations:

  $$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

6. Interval-Valued Fuzzy Sets: Mendel's Algorithm

- *Main idea behind type 2:* $m_i(x_i)$ is not a crisp number.
- *Interval-valued case:* instead of $m_i(x_i)$, we have an interval $\mathbf{m}_i(x_i) = [\underline{m}_i(x_i), \overline{m}_i(x_i)]$.
- Different $m_i(x_i) \in [\underline{m}_i(x_i), \overline{m}_i(x_i)]$ lead to different

$$m(y) = \sup\{\min(m_1(x_1), \ldots, m_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

- *Goal:* compute the range $[\underline{m}(y), \overline{m}(y)]$ of $m(y)$.
- *Idea:* the above expression is monotonic, so

$$\underline{m}(y) = \sup\{\min(\underline{m}_1(x_1), \ldots, \underline{m}_n(x_n)) : y = f(x_1, \ldots, x_n)\};$$

$$\overline{m}(y) = \sup\{\min(\overline{m}_1(x_1), \ldots, \overline{m}_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

- Using the known relation with interval computations,

$$\underline{\mathbf{y}}(\alpha) = f(\underline{\mathbf{x}}_1(\alpha), \ldots, \underline{\mathbf{x}}_n(\alpha)); \;\; \overline{\mathbf{y}}(\alpha) = f(\overline{\mathbf{x}}_1(\alpha), \ldots, \overline{\mathbf{x}}_n(\alpha)),$$

$$\underline{\mathbf{x}}_i \stackrel{\text{def}}{=} \{x_i : \underline{m}_i(x_i) \ge \alpha\} \text{ and } \overline{\mathbf{x}}_i \stackrel{\text{def}}{=} \{x_i : \overline{m}_i(x_i) \ge \alpha\}.$$

7. New Result: Extension of Mendel's Formulas to General Type-2 Fuzzy Numbers

- *Reminder:* Zadeh's extension principle

$$m(y)=\sup\{\min(m_1(x_1),m_2(x_2),\ldots):y=f(x_1,\ldots,x_n)\}.$$

- *General type-2 case:* $m_i(x_i)$ are fuzzy numbers, with β-cuts $(m_i(x_i))(\beta)=[\underline{(m_i(x_i))(\beta)},\overline{(m_i(x_i))(\beta)}]$.

- Due to known relation with interval computations:

$$(m(y))(\beta)=\sup\{\min((m_1(x_1))(\beta),\ldots):y=f(x_1,\ldots,x_n)\}.$$

- Due to monotonicity:

$$\underline{(m(y))(\beta)}=\sup\{\min(\underline{(m_1(x_1))(\beta)},\ldots):y=f(x_1,\ldots,x_n)\};$$

$$\overline{(m(y))(\beta)}=\sup\{\min(\overline{(m_1(x_1))(\beta)},\ldots):y=f(x_1,\ldots,x_n)\}.$$

- Due to known relation with interval computations:

$$\underline{\mathbf{y}}(\alpha,\beta)=f(\underline{\mathbf{x}}_1(\alpha,\beta),\ldots);\quad \overline{\mathbf{y}}(\alpha,\beta)=f(\overline{\mathbf{x}}_1(\alpha,\beta),\ldots),$$

$$\underline{\mathbf{y}}(\alpha,\beta)\stackrel{\text{def}}{=}\{x:\underline{y(\beta)}\ge\alpha\}\text{ and }\overline{\mathbf{y}}(\alpha,\beta)\stackrel{\text{def}}{=}\{x:\overline{y(\beta)}\ge\alpha\}$$

8. Conclusions

- Type-2 fuzzy sets more adequately describe expert's opinion than the more traditional type-1 fuzzy sets.
- The use of type-2 fuzzy sets has thus led to better quality control, better quality clustering, etc.
- *Main obstacle:* the computational time of data processing increases.
- *Mendel's result:* processing *interval-valued* fuzzy numbers can be reduced to interval computations.
- *Conclusion:* processing interval-valued fuzzy data is (almost) as fast as processing type-1 fuzzy data.
- In this talk, we showed that Mendel's algorithm can be extended to *general* type-2 fuzzy numbers.
- This will hopefully lead to more practical applications of type-2 fuzzy sets.

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