Towards Fast Algorithms for Processing Type-2 Fuzzy Data: Extending Mendel’s Algorithms From Interval-Valued to a More General Case

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1. Outline

• Known: processing (type-1) fuzzy data can be reduced to interval uncertainty:
  – Zadeh’s extension principle is equivalent to
  – level-by-level interval computations on $\alpha$-cuts.

• More adequate description: type-2 fuzzy sets.

• Practical limitation: transition to type-2 increases computational complexity.

• Mendel’s algorithm: for interval-valued fuzzy sets, processing can also be reduced to interval computations.

• In this talk: we show that Mendel’s ideas can be naturally extended to arbitrary type-2 fuzzy numbers.
2. Why Data Processing and Knowledge Processing Are Needed in the First Place

- **Problem:** some quantities $y$ are difficult (or impossible) to measure or estimate directly.

- **Solution:** indirect measurements or estimates

\[
\tilde{x}_1 \rightarrow \tilde{x}_2 \rightarrow \ldots \rightarrow \tilde{x}_n \rightarrow f \rightarrow \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n)
\]

- **Fact:** estimates $\tilde{x}_i$ are approximate.

- **Question:** how approximation errors $\Delta x_i \overset{\text{def}}{=} \tilde{x}_i - x_i$ affect the resulting error $\Delta y = \tilde{y} - y$?
3. From Probabilistic to Interval Uncertainty

- Manufacturers of MI provide us with bounds $\Delta_i$ on measurement errors: $|\Delta x_i| \leq \Delta_i$.
- Thus, we know that $x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$.
- Often, we also know probabilities, but in 2 cases, we don’t:
  - cutting-edge measurements;
  - cutting-cost manufacturing.
- In such situations:
  - we know the intervals $[x_i, \bar{x}_i] = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ of possible values of $x_i$, and
  - we want to find the range of possible values of $y$:
    $$y = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\}.$$
4. Main Problem of Interval Computations

We are given:

- an integer $n$;
- $n$ intervals $x_1 = [x_1, \overline{x}_1], \ldots, x_n = [x_n, \overline{x}_n]$, and
- an algorithm $f(x_1, \ldots, x_n)$ which transforms $n$ real numbers into a real number $y = f(x_1, \ldots, x_n)$.

We need to compute the endpoints $\underline{y}$ and $\overline{y}$ of the interval

$$y = [\underline{y}, \overline{y}] = \{ f(x_1, \ldots, x_n) : x_1 \in [x_1, \overline{x}_1], \ldots, [x_n, \overline{x}_n] \}.$$
5. Knowledge Processing and Fuzzy Uncertainty

- In many practical situations, we only have expert estimates for the inputs \( x_i \) like “around \( \tilde{x}_i \)”.
- Such estimates are described by membership functions \( m_i(x_i) \) or \( \alpha \)-cuts \( x_i(\alpha) = \{ x : m_i(x) \geq \alpha \} \).
- Usually, each \( m_i(x_i) \) is a fuzzy number: starts at 0, increases, then decreases to 0.
- For fuzzy numbers, \( \alpha \)-cuts are intervals.
- The membership function for \( y = f(x_1, \ldots, x_n) \) is defined by Zadeh’s extension principle:
  \[
  m(y) = \sup\{ \min(m_1(x_1), \ldots, m_n(x_n)) : y = f(x_1, \ldots, x_n) \}.
  \]
- For fuzzy numbers, computations reduce to layer-by-layer interval computations:
  \[
  y(\alpha) = f(x_1(\alpha), \ldots, x_n(\alpha)).
  \]
6. Interval-Valued Fuzzy Sets: Mendel’s Algorithm

• **Main idea behind type 2:** $m_i(x_i)$ is not a crisp number.

• **Interval-valued case:** instead of $m_i(x_i)$, we have an interval $m_i(x_i) = [\underline{m}_i(x_i), \overline{m}_i(x_i)]$.

• Different $m_i(x_i) \in [\underline{m}_i(x_i), \overline{m}_i(x_i)]$ lead to different $m(y) = \sup\{\min(m_1(x_1), \ldots, m_n(x_n)) : y = f(x_1, \ldots, x_n)\}$.

• **Goal:** compute the range $[\underline{m}(y), \overline{m}(y)]$ of $m(y)$.

• **Idea:** the above expression is monotonic, so

\[
\underline{m}(y) = \sup\{\min(\underline{m}_1(x_1), \ldots, \underline{m}_n(x_n)) : y = f(x_1, \ldots, x_n)\};
\]

\[
\overline{m}(y) = \sup\{\min(\overline{m}_1(x_1), \ldots, \overline{m}_n(x_n)) : y = f(x_1, \ldots, x_n)\}.
\]

• Using the known relation with interval computations,

\[
\underline{y}(\alpha) = f(\underline{x}_1(\alpha), \ldots, \underline{x}_n(\alpha)); \quad \overline{y}(\alpha) = f(\overline{x}_1(\alpha), \ldots, \overline{x}_n(\alpha)),
\]

\[
x_i \overset{\text{def}}{=} \{x_i : \underline{m}_i(x_i) \geq \alpha\} \quad \text{and} \quad \overline{x}_i \overset{\text{def}}{=} \{x_i : \overline{m}_i(x_i) \geq \alpha\}.
\]
7. **New Result: Extension of Mendel’s Formulas to General Type-2 Fuzzy Numbers**

- **Reminder:** Zadeh’s extension principle
  \[ m(y) = \sup\{\min(m_1(x_1), m_2(x_2), \ldots) : y = f(x_1, \ldots, x_n)\}. \]

- **General type-2 case:** \( m_i(x_i) \) are fuzzy numbers, with \( \beta \)-cuts \( (m_i(x_i))(\beta) = [(m_i(x_i))(\beta), (m_i(x_i))(\beta)] \).

- Due to known relation with interval computations:
  \[ (m(y))(\beta) = \sup\{\min((m_1(x_1))(\beta), \ldots) : y = f(x_1, \ldots, x_n)\}. \]

- Due to monotonicity:
  \[ (m(y))(\beta) = \sup\{\min((m_1(x_1))(\beta), \ldots) : y = f(x_1, \ldots, x_n)\}; \]
  \[ (m(y))(\beta) = \sup\{\min((m_1(x_1))(\beta), \ldots) : y = f(x_1, \ldots, x_n)\}. \]

- Due to known relation with interval computations:
  \[ y(\alpha, \beta) = f(x_1(\alpha, \beta), \ldots); \quad \overline{y}(\alpha, \beta) = f(\overline{x}_1(\alpha, \beta), \ldots), \]
  \[ \underline{y}(\alpha, \beta) \overset{\text{def}}{=} \{x : y(\beta) \geq \alpha\} \text{ and } \overline{y}(\alpha, \beta) \overset{\text{def}}{=} \{x : \overline{y}(\beta) \geq \alpha\} \]
8. Conclusions

- Type-2 fuzzy sets more adequately describe expert’s opinion than the more traditional type-1 fuzzy sets.
- The use of type-2 fuzzy sets has thus led to better quality control, better quality clustering, etc.
- Main obstacle: the computational time of data processing increases.
- Mendel’s result: processing interval-valued fuzzy numbers can be reduced to interval computations.
- Conclusion: processing interval-valued fuzzy data is (almost) as fast as processing type-1 fuzzy data.
- In this talk, we showed that Mendel’s algorithm can be extended to general type-2 fuzzy numbers.
- This will hopefully lead to more practical applications of type-2 fuzzy sets.
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