Least Sensitive (Most Robust) Fuzzy "Exclusive Or" Operations

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1. Need for Fuzzy "Exclusive Or" Operations

- One of the main objectives of fuzzy logic is to formalize commonsense and expert reasoning.
- People use logical connectives like "and" and "or".
- Commonsense "or" can mean both "inclusive or" and "exclusive or".
- *Example:* A vending machine can produce either a coke or a diet coke, but not both.
- In mathematics and computer science, "inclusive or" is the one most frequently used as a basic operation.
- *Fact:* "Exclusive or" is also used in commonsense and expert reasoning.
- *Thus:* There is a practical need for a fuzzy version.
- *Comment:* "exclusive or" is actively used in computer design and in quantum computing algorithms

2. A Crisp "Exclusive Or" Operation: A Reminder

- *Fuzzy analogue* of a classical logic operation op:
  - we know the experts' degree of belief $a = d(A)$ and $b = d(B)$ in statements A and B;
  - based on a and b, we want to estimate the degree of belief in "A op B", as $f_{\rm op}(a, b)$.
- For op $= \&$, we get an "and"-operation (t-norm).
- For op $= \vee$, we get an "or"-operation (t-conorm).
- As usual, the fuzzy "exclusive or" operation must be an extension of the corresponding crisp operation $\oplus$.
- In the traditional 2-valued logic, $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$.
- Thus, the desired fuzzy "exclusive or" operation $f_\oplus(a, b)$ must satisfy the same properties:

$$f_\oplus(0, 0) = f_\oplus(1, 1) = 0; \quad f_\oplus(0, 1) = f_\oplus(1, 0) = 1.$$

3. Need for the Least Sensitivity: Reminder

- One of the main ways to elicit degree of certainty d is to ask to pick a value on a scale. Example:
  - on a scale of 0 to 10, an expert picks 8, so we get
  $$d = 8/10 = 0.8;$$
  - on a scale from 0 to 8, whatever we pick, we cannot get 0.8: $6/8 = 0.75 < 0.8$; $7/8 = 0.875 > 0.8$.
  - the expert will probably pick 6, with
  $$d' = 6/8 = 0.75 \approx 0.8.$$
- *It is desirable:* that the result of the fuzzy operation not change much if we slightly change the inputs:
$$|f(a,b) - f(a',b')| \le k \cdot \max(|a - a'|, |b - b'|),$$
with the smallest possible k.
- Such operations are called *the least sensitive* or *the most robust.*

4. For t-Norms and t-Conorms, the Least Sensitivity Requirement Leads to Reasonable Operations

- *Known results:*
  - There is only one least sensitive t-norm ("and"-operation)
$$f_\&(a,b)=\min(a,b).$$
  - There is also only one least sensitive t-conorm ("or"-operation)
$$f_\vee(a,b)=\max(a,b).$$
- *What we do in this presentation:* we describe the least sensitive fuzzy "exclusive or" operation.

5. Definition of a Fuzzy Exclusive-Or Operation

- **Definition:** *A function* $f : [0,1] \times [0,1] \to [0,1]$ *is called a* fuzzy "exclusive or" operation *if*

$$f(0,0) = f(1,1) = 0 \text{ and } f(0,1) = f(1,0) = 1.$$

- *Comment:* We could also require other conditions, e.g., commutativity and associativity.
- However, our main objective is to select a single operation which is the least sensitive.
- *Fact:* The weaker the condition, the larger the class of operations that satisfy these conditions.
- *Thus:* the stronger the result that our operation is the least sensitive in this class.
- *Conclusion:* We select the weakest possible condition to make our result as strong as possible.

6. Main Result

Definition:

- *Let F be a class of functions from* $[0,1]\times[0,1]$ *to* $[0,1]$.
- *We say that a function $f \in F$ is* the least sensitive *in the class F if it satisfies the following two conditions:*
  - *for some real number k, the function f satisfies the condition*

    $$|f(a,b)-f(a',b')| \le k\cdot \max(|a-a'|,|b-b'|);$$

  - *no other function $f \in F$ satisfies this condition.*

Theorem: *In the class of all fuzzy "exclusive or" operations, the following function is the least sensitive:*

$$f_\oplus(a,b)=\min(\max(a,b),\max(1-a,1-b)).$$

7. Interpretation of the Main Result

- *Reminder:* the least sensitive operation is

$$f_\oplus(a,b) = \min(\max(a,b), \max(1-a, 1-b)).$$

- *Fact:* in 2-valued logic, "exclusive or" $\oplus$ can be described in terms of the "inclusive or" operation $\vee$ as

$$a \oplus b \Leftrightarrow (a \vee b) \,\&\, \neg(a \,\&\, b).$$

- *Natural idea:*
  - replace $\vee$ with the least sensitive "or"-operation $f_\vee(a,b) = \max(a,b)$,
  - replace & with the least sensitive "and"-operation $f_\&(a,b) = \min(a,b)$, and
  - replace $\neg$ with the least sensitive negation operation $f_\neg(a) = 1 - a$,
- *Result:* we get the expression given in the Theorem.

8. Proof of the Main Result: 1st Condition

- *Reminder:* $f_\oplus(a,b) = \min(\max(a,b), \max(1-a, 1-b))$.
- *We need to prove* the following two conditions:
  - *1st:* that this function $f_\oplus(a,b)$ satisfies the following condition with $k = 1$:

$$|f(a,b) - f(a',b')| \le k \cdot \max(|a-a'|, |b-b'|);$$

  - *2nd:* that no other "exclusive or" operation satisfies this property.
- *1st condition:* let us prove that for every $\varepsilon > 0$, if $|a - a'| \le \varepsilon$ and $|b - b'| \le \varepsilon$, then

$$|f_\oplus(a,b) - f_\oplus(a',b')| \le \varepsilon.$$

- *It is known:* that the functions $\min(a,b)$, $\max(a,b)$, and $1 - a$ satisfy the above condition with $k = 1$.

9. Proof of the Main Result (cont-d)

- *Known results:* if $|a-a'|\le \varepsilon$ and $|b-b'|\le \varepsilon$, then the following three inequalities hold:

$$|\max(a,b)-\max(a',b')|\le \varepsilon;$$

$$|(1-a)-(1-a')|\le \varepsilon; \text{ and } |(1-b)-(1-b')|\le \varepsilon.$$

- From the result above, by using the condition for the max operation, we conclude that

$$|\max(1-a,1-b)-\max(1-a',1-b')|\le \varepsilon.$$

- Now, from the results above, by using the condition for the min operation, we conclude that

$$|\min(\max(a,b),\max(1-a,1-b))$$
$$-\min(\max(a',b'),\max(1-a',1-b'))|\le \varepsilon.$$

- The statement is proven.

10. Fuzzy "Exclusive Or" Operations $f(a,b)$ Which Are the Least Sensitive on Average

- *Idea:* select f so that *on average*, the change in a and b leads to the smallest possible change Δc in $c = f(a,b)$.
- *Assumption:* Δa and Δb are independent random variables with 0 mean and small variance σ^2.
- *Objective:* estimate $\Delta c = f(a+\Delta a, b+\Delta b) - f(a,b)$.
- Since Δa and Δb are small, we can keep only linear terms in the Taylor series of Δc w.r.t. Δa and Δb:
$$\Delta c \approx \frac{\partial f}{\partial a} \cdot \Delta a + \frac{\partial f}{\partial b} \cdot \Delta b.$$
- Since the variables are independent with 0 mean, the mean of Δc is also 0, and variance of Δc is equal to
$$\sigma^2(a,b) = \left(\left(\frac{\partial f}{\partial a} \right)^2 + \left(\frac{\partial f}{\partial b} \right)^2 \right) \cdot \sigma^2.$$

11. Fuzzy "Exclusive Or" Operations Which Are the Least Sensitive on Average (cont-d)

- *Reminder:* for each a and b, the variance $\sigma^2(a,b)$ of Δc is equal to

$$\sigma^2(a,b)=\left(\left(\frac{\partial f}{\partial a}\right)^2+\left(\frac{\partial f}{\partial b}\right)^2\right)\cdot\sigma^2.$$

- To get the "average" variance, it is reasonable to average this value $\sigma^2(a,b)$ over all possible a and b.
- *Resulting average value:* $I\cdot\sigma^2$, where

$$I\stackrel{\text{def}}{=}\int_{a=0}^{a=1}\int_{b=0}^{b=1}\left(\left(\frac{\partial f}{\partial a}\right)^2+\left(\frac{\partial f}{\partial b}\right)^2\right)\,da\,db.$$

- *We want:* the average sensitivity to be the smallest.
- *Conclusion:* we select the function $f(a,b)$ for which the integral I takes the smallest possible value.

12. New Result: Formulation

- *Reminder:* we consider *"exclusive or"* operations $f(a,b)$, i.e., functions $f:[0,1]\times[0,1]\to[0,1]$ for which:

$$f(0,b)=b,\ \ f(a,0)=a,\ \ f(1,b)=1-b,\ \text{and}\ f(a,1)=1-a.$$

- *Main result:* among all such operations, the operation which is the least sensitive on average has the form

$$f_{\oplus}(a,b)=a+b-2\cdot a\cdot b.$$

- *Interpretation:*
  - the classical (2-valued) "exclusive or" operation $a\oplus b$ can be represented as $(a\vee b)\,\&\,(\neg a\vee\neg b)$;
  - use the fuzzy analogues of $\&$, $\vee$, and $\neg$ which are the least sensitive on average:

$$f_{\&}(a,b)=\max(p+q-1,0);\ \ f_{\vee}(a,b)=p+q-p\cdot q;$$

$$f_{\neg}(a)=1-a.$$