Fusing Continuous and Discrete Data, on the Example of Merging Seismic and Gravity Models in Geophysics

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1. Case Study

- To find the density $\rho(v)$ at different locations and different depths, we can use two types of data:
  - the \textit{seismic data}, i.e., the arrival times of signals from earthquake and test explosions;
  - the \textit{gravity data}.

- Both data provide complementary information:
  - seismic data provides information about a narrow zone around a path;
  - gravity data provides information about the larger area – but with much smaller spatial resolution.

- At present, there are no efficient algorithms for processing both types of data.

- So, we must fuse the results of processing these two types of data: a seismic model and a gravity model.
2. Computational Problem: Need to Fuse Discrete and Continuous Models

- Traditionally, seismic models are *continuous*: the velocity smoothly changes with depth.
- In contrast, the gravity models are *discrete*: we have layers, in each of which the velocity is constant.
- The abrupt transition corresponds to a steep change in the continuous model.
- Both models locate the transition only approximately.
- So, if we simply combine the corresponding values value-by-value, we will have *two* transitions instead of one:
  - one transition where the continuous model has it, and
  - another transition nearby where the discrete model has it.
3. What We Plan to Do

- *We want* to avoid the misleading double-transition models.

- *Idea:* first fuse the corresponding transition locations.

- *In this paper,* we provide an algorithm for such location fusion.

- *Specifically,* first, we formulate the problem in the probabilistic terms.

- *Then,* we provide an algorithm that produces the most probable transition location.

- *We show* that the result of the probabilistic location algorithm is in good accordance with common sense.

- *We also show* how the commonsense intuition can be reformulated in fuzzy terms.
4. Available Data: What is Known and What Needs to Be Determined

- For each location, in the discrete model, we have the exact depth $z_d$ of the transition.
- In contrast, for the continuous model, we do not have the abrupt transition.
- Instead, we have velocity values $v(z)$ at different depths.
- We must therefore extract the corresponding transition value $z_c$ from the velocity values.
- To be more precise, we have values $v_1, v_2, \ldots, v_i, \ldots, v_n$ corresponding to different depths.
- We need to find $i$ for which the transition occurs between the depths $i$ and $i + 1$. 
5. Probabilistic Approach

- The difference $\Delta v_j \overset{\text{def}}{=} v_j - v_{j+1}$ $(j \neq i)$ is caused by many independent factors.

- Due to the Central Limit Theorem, we thus assume that it is normally distributed, with probability density

$$p_j \overset{\text{def}}{=} \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma}} \cdot \exp \left( -\frac{1}{2 \cdot \sigma^2} \cdot (\Delta v_j)^2 \right).$$

- The value $\Delta v_i$ at the transition depth $i$ is \textit{not} described by the normal distribution.

- We assume that differences corresponding to different depths $j$ are independent, so:

$$L_i = \prod_{j \neq i} p_j = \prod_{j \neq i} \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma}} \cdot \exp \left( -\frac{1}{2 \cdot \sigma^2} \cdot (\Delta v_j)^2 \right).$$
6. How to Find the Location: The General Idea of the Maximum Likelihood Approach

- *Reminder:* the likelihood of each model is:

\[
L_i = \prod_{j \neq i} p_j = \prod_{j \neq i} \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma}} \cdot \exp \left( -\frac{1}{2 \cdot \sigma^2} \cdot (\Delta v_j)^2 \right).
\]

- *Natural idea:* select the parameters for which the likelihood of the observed data is the largest.

- The value \( L_i \) is the largest if and only if \(- \ln(L_i)\) is the smallest:

\[
- \ln(L_i) = \text{const} + \frac{1}{2 \cdot \sigma^2} \cdot \sum_{j \neq i}(\Delta v_j)^2 \to \min_i.
\]

- This sum is equal to \( \sum_{j \neq i}(\Delta v_j)^2 = \sum_{j=1}^{n-1}(\Delta v_j)^2 - (\Delta v_i)^2 \).

- The first term in this expression does not depend on \( i \).

- Thus, the difference is the smallest \( \iff \) the value \( (\Delta v_i)^2 \) is the largest \( \iff |\Delta v_i| \) is the largest.
7. Resulting Location

- **We want:** to select the most probable location of the transition point.

- **We select:** the depth $i_0$ for which the absolute value $|\Delta v_i|$ of the difference $\Delta v_i = v_{i+1} - v_i$ is the largest.

- This conclusion seems to be very reasonable:
  
  - the most probable location of the actual abrupt transition between the layers
  
  - is the depth at which the measured difference is the largest.
8. The Results of the Probabilistic Approach are in Good Accordance with Common Sense

- Intuitively, for each depth $i$, our confidence that $i$ is a transition point depends on the difference $|\Delta v_i|$:
  - the smaller the difference, the less confident we are that this is the actual transition depth, and
  - the larger the difference, the more confident we are that this is the actual transition depth.

- In our probabilistic model, we select a location with the largest possible value $|\Delta v_i|$.

- This shows that the probabilistic model is in good accordance with common sense.

- This coincidence increases our confidence in this result.
9. It May Be Useful to Formulate the Common Sense Description in Fuzzy Terms

- Fuzzy logic is known to be a useful way to formalize imprecise commonsense reasoning.
- Common sense: the degree of confidence $d_i$ that $i$ is a transition point is $f(|\Delta v_i|)$, for some monotonic $f(z)$.
- It is reasonable to select a value $i$ for which our degree of confidence is the largest $d_i = f(|\Delta v_i|) \to \text{max}$.
- Since $f(z)$ is increasing, this is equivalent to $|\Delta v_i| \to \text{max}$.
- Of course, to come up with this conclusion, we do not need to use the fuzzy logic techniques.
- However, this description may be useful if we also have other expert information.
10. How Accurate Is This Location Estimate?

- **Reminder:** the likelihood has the form
  \[ L_i = \prod_{j \neq i} p_j = \prod_{j \neq i} \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma}} \cdot \exp \left( -\frac{1}{2 \cdot \sigma^2} \cdot (\Delta v_j)^2 \right). \]

- **We have found** the most probable transition \( i_0 \) as the value for which \( L_i \) is the largest.

- **Similarly:** we can find \( \sigma \) for which \( L_i \) is the largest:
  \[ \sigma^2 = \frac{1}{n - 2} \cdot \sum_{j \neq i_0} (\Delta v_j)^2. \]

- The probability \( P_i \) that the transition is at location \( i \) is proportional to \( L_i \): \( P_i = c \cdot L_i \).

- The coefficient \( c \) can be determined from the condition that the total probability is 1: \( 1 = \sum_i P_i = c \cdot \sum_{i=1}^n L_i. \)

- So, \( c = (\sum L_i)^{-1}. \)
11. Accuracy of the Location Estimate (cont-d)

- The mean square deviation $\sigma_0^2$ of the actual transition depth from our estimate $i_0$ is defined as

$$\sigma_0^2 = \sum_{i=1}^{n-1} (i - i_0)^2 \cdot P_i.$$

- We know that $P_i = c \cdot L_i$, and we have formulas for computing $L_i$ and $c$, so we can compute $\sigma_0$.

- We applied this algorithm to the seismic model of El Paso area, and got $\sigma_0 \approx 1.5$ km.

- This value is of the same order (1-2 km) as the difference between:
  - the border depth estimates coming from the seismic data and
  - the border depth coming from the gravity data.
12. How to Fuse the Depth Estimates

- Now, we have two estimates for the transition depth:
  - the estimate $i_d$ from the discrete (gravity) model;
  - the estimate $i_0$ from the continuous (seismic) model.
- The estimate $i_d$ comes from a standard statistical analysis, so we know standard deviation $\sigma_d$.
- For $i_0$, we already know the standard deviation $\sigma_0$.
- It is reasonable to assume that both differences $i_d - i$ and $i_0 - i$ are normally distributed and independent:
  $$p_i = \exp \left( -\frac{(i_d - i_f)^2}{2 \cdot \sigma_d^2} \right) \cdot \exp \left( -\frac{(i_0 - i_f)^2}{2 \cdot \sigma_0^2} \right).$$
- The most probable location $i$ is when $p_i \to \max$, i.e.:
  $$i_f = \frac{i_d \cdot \sigma_d^{-2} + i_0 \cdot \sigma_0^{-2}}{\sigma_d^{-2} + \sigma_0^{-2}}.$$
13. Towards Fusing Actual Maps

- In the discrete model:
  - values $i < i_d$ correspond to the upper zone;
  - values $i > i_d$ correspond to the lower zone.

- Similarly, in the continuous model:
  - values $i < i_0$ correspond to the upper zone;
  - values $i > i_0$ correspond to the lower zone.

- So, for depths $i \leq \min(i_0, i_d)$ and $i \geq \max(i_0, i_d)$, both models correctly describe the zone.

- For these depths, we can simply fuse the values from both models.

- We can fuse them similarly to how we fused the depths.

- For intermediate depths, we need to adjust the models: e.g., by taking the nearest value from the correct zone.
14. How to Fuse the Actual Maps: First Stage

- **First**: we adjust both models so that they both have a transition at depth $i_f$.

- **Adjusting the discrete model** is easy: we replace
  - the original depth $i_d$
  - with the new (more accurate) fused value $i_f$.

- **Adjusting the continuous model**:
  - when $i_f < i_0$, the values at depths $i$ between $i_f$ and $i_0$ are erroneously assigned to the the upper zone;
  - these values $v_i$ must be replaced by the the value of the nearest point at the lower zone $v_{i_0+1}$;
  - when $i_f > i_0$, the values at depths $i$ between $i_0$ and $i_f$ are erroneously assigned to the the lower zone;
  - these values $v_i$ must be replaced by the the value of the nearest point at the upper zone $v_{i_0}$. 
15. How to Merge the Adjusted Models

- For each depth $i$, we now have two adjusted values $v'_i$ and $v''_i$ corresponding to two adjusted models.
- Let $\sigma'$ and $\sigma''$ be the corresponding standard deviations.
- It is reasonable to assume that both differences $v'_i - v_i$ and $v''_i - v_i$ are normally distributed and independent:
  \[ p(v_i) = \exp \left( -\frac{(v'_i - v_i)^2}{2 \cdot (\sigma')^2} \right) \cdot \exp \left( -\frac{(v''_i - v_i)^2}{2 \cdot (\sigma'')^2} \right). \]
- The most probable value $\tilde{v}_i$ is when $p(v_i) \to \max$, i.e.:
  \[ \tilde{v}_i = \frac{v'_i \cdot (\sigma')^{-2} + v''_i \cdot (\sigma'')^{-2}}{(\sigma')^{-2} + (\sigma'')^{-2}}. \]
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