From 1-D to 2-D Fuzzy: A Proof that Interval-Valued and Complex-Valued Are the Only Distributive Options

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1. Fuzzy Logic: Reminder

- In the traditional two-valued logic, every statement is either true or false.
- In the computer these values are represented as, correspondingly 1 and 0.
- These two values cannot capture a situation when an expert is not 100% sure about his/her statement.
- To capture such expert uncertainty, L. Zadeh came up with an idea of fuzzy logic, where for each statement:
  - instead of two possible truth values 0 and 1,
  - we can have degrees of certainty that can take any values from 0 to 1.
- We now need to extend propositional operations from \{0, 1\} to [0, 1].
2. Fuzzy Logic (cont-d)

- We need to extend propositional operations from \{0, 1\} to [0, 1].

- From the purely mathematical viewpoint, there are many such extensions.

- It is desirable to preserve as many properties of the 2-valued logic as possible.

- Usually, “and”- and “or”-operations are selected to be commutative and associative.

- This still leaves us with plenty of different choices.

- It is therefore desirable, among all such operations, to select those that satisfy additional properties.
3. Distributivity

- One of such additional natural properties is distributivity, that $A \& (B \lor C)$ is equivalent to $(A \& B) \lor (A \& C)$:
  \[
  f_\&(a, f_\lor(b, c)) = f_\lor(f_\&(a, b), f_\&(a, c)).
  \]

- If we require this for all $a$, $b$, and $c$, then
  \[
  f_\lor(a, b) = \max(a, b).
  \]

- It is known that sometimes, the expert’s use of “or” is better described by other “or”-operations.

- It is reasonable to restrict the above equality to cases when $f_\lor(b, c) < 1$.

- Then, “and”- and “or”-operations are equivalent to $f_\&(a, b) = a \cdot b$ and $f_\lor(a, b) = \min(a + b, 1)$. 
4. Need to Go Beyond $[0, 1]$

- The $[0, 1]$-based fuzzy logic captures many features of expert uncertainty.

- However, in some situations, it is not fully adequate to distinguish between different situations; e.g.:
  - if we have no information about a given statement,
  - then it makes sense to describe this uncertainty by the midpoint 0.5.

- On the other hand:
  - if have exactly as many arguments supporting $S$ as supporting $\neg S$,
  - then it also makes sense to describe this uncertainty by the value 0.5.

- In both situations, the truth value is the same, but the uncertainty is different.
5. Need for 2-D Extensions

- If we add an argument in support of \( S \), then:
  - in the first case, we now have an argument supporting \( S \) and no arguments supporting \( \neg S \),
  - so the truth value of \( S \) should drastically increase;
  - in the second case, the numbers of statement supporting \( S \) and \( \neg S \) remains almost equal;
  - so, the truth value should not change much.

- To distinguish between such situations, it is desirable:
  - to supplement the \([0, 1]\)-valued degree of belief
  - with an additional number (or numbers).

- The simplest case: use one additional number.

- Thus, we use two numbers to describe our degree of certainty in a given statement.
6. 2-D Extensions Should Be Commutative, Associative, and Distributive

- From the commonsense viewpoint, logical operations are commutative, associative, and distributive.
- It is thus reasonable to require that the 2-D extensions of satisfy these three properties.
- The most widely used 2-D extension is interval-valued fuzzy logic.
- There, our degree of certainty in a statement is described by an interval $[\underline{d}, \overline{d}] \subseteq [0, 1]$.
- This enables us to clearly distinguish between the above two situations:
  - the case of complete uncertainty is naturally described by the interval $[0, 1]$, while
  - the case when equally many arguments for $S$ and for $\neg S$ is described by $[0, 5, 0.5] = \{0.5\}$. 
7. Distributive 2-D Extensions

- In principle, we can extend different t-norms and t-conorms to the interval-valued case:
  \[ f([a, \overline{a}], [b, \overline{b}]) \overset{\text{def}}{=} \{ f(a, b) : a \in [a, \overline{a}] \text{ and } b \in [b, \overline{b}] \}. \]

- In particular, for \( a \cdot b \) and \( a + b \), we get
  \[ [a, \overline{a}] \cdot [b, \overline{b}] = [a \cdot b, \overline{a} \cdot \overline{b}]; \quad [a, \overline{a}] + [b, \overline{b}] = [a + b, \overline{a} + \overline{b}]. \]
- The resulting interval-valued logic is distributive.
- Another useful 2-D distributive extension of the usual fuzzy logic is the complex-valued fuzzy logic.
- In this logic, degrees of belief can take any complex values \( a + b \cdot i \), with \( i \overset{\text{def}}{=} \sqrt{-1} \).
- The complex-valued logic lacks a clear justification and clear interpretation.
- Thus, it is not as widely used an interval-valued one.
8. Are There Other Extension?

- At first glance, it looks like:
  - the above two extensions have been rather arbitrarily chosen, and
  - in principle, there are many other extensions.

- We show that interval-valued and complex-valued are the only possible 2-D distributive extensions.

- This result elevates complex-valued fuzzy logic:
  - from the status of one of the mathematically possible extensions
  - to a much higher status of one of the two possible extensions.

- This will, hopefully lead to a more frequent use of complex-valued fuzzy logic.
9. 2-D Logic: Set of Possible Values

- Let \( \odot \) and \( \oplus \) be 2-D extensions of \( \cdot \) and \( + \).
- Let \( x \) be a 2-D element different from real numbers.
- On this extended set, we want to allow multiplication.
- Thus, we need to consider elements of the type \( b \odot x \) for arbitrary real numbers \( b \).
- We also want to allow addition between real numbers \( a \) and the products \( b \odot x \): \( a \oplus (b \odot x) \).
- The set of all such elements depends on two parameters \( a \) and \( b \) and is, thus, 2-dimensional.
- We are interested in 2-D extensions.
- Thus, the desired extension cannot contain any other elements.
- So, each extension is the set of all the elements of the type \( a \oplus (b \odot x) \).
10. Addition ("Or"-Operation) on the Set of Possible Values

- Due to commutativity and associativity of $\oplus$, we get
\[(a \oplus (b \odot x)) \oplus (a' \oplus (b' \odot x)) = (a \oplus a') \oplus ((b \odot x) \oplus (b' \odot x)).\]

- Here, $a$ and $a'$ are both real numbers, so
\[(a \oplus (b \odot x)) \oplus (a' \oplus (b' \odot x)) = (a + a') \oplus ((b \odot x) \oplus (b' \odot x)).\]

- Distributivity implies $(b \odot x) \oplus (b' \odot x) = (b \oplus b') \cdot x,$
so $(b \odot x) \oplus (b' \odot x) = (b + b') \odot x.$

- Substituting this expression into the above formula for
\[(a \oplus (b \odot x)) \oplus (a' \oplus (b' \odot x)),\]
we get
\[(a \oplus (b \odot x)) \oplus (a' \oplus (b' \odot x)) = (a + a') \oplus ((b + b') \odot x).\]

- In other words, we have a component-wise addition.
11. Multiplication (“And”-Operation) on the Set of Possible Values

- Due to distributivity, we have
  \[
  (a \oplus (b \odot x)) \odot (a' \oplus (b' \odot x)) = (a \odot a') \oplus ((a \odot b' + a' \odot b) \odot x) \oplus ((b \odot b') \odot (x \odot x)).
  \]

- Since for real numbers, the new operations \( \odot \) and \( \oplus \) are simply multiplication and addition, we get:
  \[
  (a \oplus (b \odot x)) \odot (a' \oplus (b' \odot x)) = (a \cdot a') \oplus ((a \cdot b' + a' \cdot b) \odot x) \oplus ((b \cdot b') \odot (x \odot x)).
  \]

- Thus, to describe the product of the new objects, it is sufficient to know the value of \( x \odot x \).

- Since all the new elements have the form \( a \oplus (b \odot x) \), we thus have \( x \odot x = p \oplus (q \odot x) \) for some \( p \) and \( q \).
12. Multiplication Simplified

- let us show that we can simplify the formula for \( x \odot x \) by re-selecting the element \( x \).

- First, instead of \( x \), we can select \( x' = x \oplus \left(-\frac{q}{2}\right) \); then, 
\[
x' \odot x' = p', \quad \text{where } p' \overset{\text{def}}{=} p + \frac{q^2}{4}.
\]

- Thus, without losing generality, we can assume that 
\( x \odot x = p \) for some real number \( p \).

- We will consider 3 cases: \( p > 0 \), \( p < 0 \), and \( p = 0 \).

- When \( p > 0 \), we can simplify the above formula even more, by considering 
\( x'' = \frac{1}{\sqrt{p}} \odot x \); then, 
\[
x'' \odot x'' = 1.
\]

- If we interpret \( a \oplus (b \odot x'') \) as \( [a - b, a + b] \), then \( \oplus \) and 
\( \odot \) become interval addition and multiplication.
13. Cases When \( p < 0 \) and When \( p = 0 \)

- When \( p < 0 \), we can take \( x'' = \frac{1}{\sqrt{|p|}} \odot x \), then
  \[ x'' \odot x'' = -1, \]
  and we get complex-valued fuzzy logic.
- When \( p = 0 \), we get
  \[
  (a \oplus (b \odot x)) \odot (a' \oplus (b' \odot x)) = (a \cdot a') \oplus ((a \cdot b' + a' \cdot b) \odot x).
  \]
- Let us show that this formula corresponds to linearized approach to uncertainty.
- We are interested in a quantity \( y \) which depend on the directly measured quantities \( x_1, \ldots, x_n \) as
  \[
  y = f(x_1, \ldots, x_n).
  \]
- We use the results \( \tilde{x}_i \) of measuring \( x_i \) to estimate \( y \) as
  \[
  \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n).
  \]
14. Case When \( p = 0 \) (cont-d)

- We assume that measurement results \( \tilde{x}_i \) are reasonably accurate.

- So, we can safely ignore the terms that are quadratic (or higher order) in terms of the measurement errors

\[
\Delta x_i \quad \text{def} = \tilde{x}_i - x_i.
\]

- Thus, \( y = \tilde{y} + \sum_{i=1}^{n} y_i \cdot \Delta x_i \), where

\[
y_i \quad \text{def} = -\frac{\partial f}{\partial x_i}.
\]

- If we have a second quantity \( y' = \tilde{y}' + \sum_{i=1}^{n} y'_i \cdot \Delta x_i \), then their sum and product have the form

\[
y + y' = (\tilde{y} + \tilde{y}') + \sum_{i}(y_i + y'_i) \cdot \Delta x_i;
\]

\[
y \cdot y' = (\tilde{y} \cdot \tilde{y}') + \sum_{i}(\tilde{y} \cdot y'_i + \tilde{y}' \cdot y_i) \cdot \Delta x_i.
\]
15. Case When $p = 0$: Conclusion

- In particular, for $n = 1$, we have exactly the above formulas corresponding to $p = 0$:

\[ y + y' = (\tilde{y} + \tilde{y}') + (y_1 + y'_1) \cdot \Delta x_1; \]
\[ y \cdot y' = (\tilde{y} \cdot \tilde{y}') + (\tilde{y} \cdot y'_1 + \tilde{y}' \cdot y_1) \cdot \Delta x_1. \]

- Thus, the case $p = 0$ corresponds to a special case of interval-valued fuzzy logic:
  - when intervals are narrow
  - so that we can ignore terms which are quadratic in terms of their width.
16. General Conclusion

We have thus shown that there are only two types of distributive 2-D fuzzy logic:

- interval-valued or
- complex-valued.
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