Sometimes, It Is Beneficial to Process Different Types of Uncertainty Separately

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1. Need for Data Processing

- One of the main objectives of science is to predict future values \( y \) of physical quantities:
  - in meteorology, we need to predict future weather;
  - in airplane control, we need to predict the location and the velocity of the plane under current control.

- To make this prediction:
  - we need to know the relation \( y = f(x_1, \ldots, x_n) \) between \( y \) and related quantities \( x_1, \ldots, x_n \);
  - then, we measure or estimate \( x_1, \ldots, x_n \);
  - finally, we use the results \( \tilde{x}_i \) of measurement (or estimation) to compute an estimate
    \[
    \tilde{y} = f(\tilde{x}_1, \ldots, \tilde{x}_n).
    \]

- This computation of \( \tilde{y} \) is an important case of data processing.
2. Need to Take Uncertainty into Account

- Measurements are never absolutely accurate, and expert estimates are even less accurate.

- As a result, the estimates $\tilde{x}_i$ are, in general, different from the actual (unknown) values $x_i$.

- Therefore, the estimate $\tilde{y}$ is also only approximate.

- In practice, it is desirable to know how accurate is this estimate $\tilde{y}$.

- To find this out, we need to take into account the accuracy of the estimates $\tilde{x}_i$.

- For measurements, we usually know the upper bound $\Delta_i$ on the absolute value of the measurement error:

  $$|\Delta x_i| \leq \Delta_i,$$

  where $\Delta x_i \overset{\text{def}}{=} \tilde{x}_i - x_i$.

- The upper bound $\Delta_i$ is usually provided by the manufacturer of the measurement instrument.
3. Taking Uncertainty into Account (cont-d)

- Once we know $\Delta_i$ and $\tilde{x}_i$, then we know that the actual value $x_i$ is located in the interval $[\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$.

- To gauge the accuracy of expert estimates, it is reasonable to use fuzzy techniques, i.e., to describe:
  - for each possible value $x_i$,
  - the degree $\mu_i(x_i)$ to which $x_i$ is possible.

- Sometimes, we also know the probabilities of different $\Delta x_i \in [-\Delta_i, \Delta_i]$; we plan to analyze this in the future.

- The prediction model is often approximate:
  \[ y = f(x_1, \ldots, x_n) + \Delta m, \text{ with } \Delta m \neq 0. \]

- Sometimes, we know the upper bound $\Delta_m$ on the model inaccuracy $\Delta m$: $|\Delta m| \leq \Delta_m$.

- In other cases, we know a membership function $\mu_m(\Delta m)$ that describes $\Delta m$. 
4. Measurement and Estimation Inaccuracies Are Usually Small

• In many practical situations, the measurement and estimation inaccuracies $\Delta x_i$ are relatively small.

• Then, we can safely ignore terms which are quadratic (or of higher order) in terms of $\Delta x_i$:

$$\Delta y = \tilde{y} - y = f(\tilde{x}_1, \ldots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \ldots, \tilde{x}_n - \Delta x_n) - \Delta m =$$

$$\sum_{i=1}^{n} c_i \cdot \Delta x_i - \Delta m, \text{ where } c_i = \frac{\partial f}{\partial x_i}.$$

• If needed, the derivative can be estimated by numerical differentiation

$$c_i \approx \frac{f(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_i + h, \tilde{x}_{i+1}, \ldots, \tilde{x}_n) - \tilde{y}}{h}.$$
5. Estimating Accuracy of Data Processing

- The value \( \Delta y = \sum_{i=1}^{n} c_i \cdot \Delta x_i - \Delta m \) is the largest when each term is the largest, so \( \Delta = \sum_{i=1}^{n} |c_i| \cdot \Delta_i + \Delta_m \).

- In the fuzzy case, the similar formula holds for the \( \alpha \)-cuts, for every \( \alpha \): \( \alpha \Delta = \sum_{i=1}^{n} |c_i| \cdot \alpha \Delta_i + \alpha \Delta_m \).

- Experts cannot describe their degrees of confidence \( \alpha \) with too much accuracy.

- Usually, it is sufficient to consider only eleven values \( \alpha = 0.0, \alpha = 0.1, \alpha = 0.2, \ldots, \alpha = 0.9, \text{ and } \alpha = 1.0 \).

- Thus, we need to apply the above formula eleven times.

- This is in line with the fact we usually divide each quantity into \( 7 \pm 2 \) categories (Miller’s “7 ± 2 Law”).

- So, it is sufficient to have at least 9 different categories.
6. Cases for Which Simplification Is Possible

- Sometimes, all membership functions are “of the same type”: \( \mu(z) = \mu_0(k \cdot z) \) for some symmetric \( \mu_0(z) \).

- Example: for triangular functions,
  \[
  \mu_0(z) = \max(1 - |z|, 0).
  \]

- In this case, \( \mu(z) \geq \alpha \) is equivalent to \( \mu_0(k \cdot z) \geq \alpha \), so \( \alpha \Delta_0 = k \cdot \alpha \Delta \) and \( 0 \Delta_0 = k \cdot 0 \Delta \).

- Thus, \( \alpha \Delta = f(\alpha) \cdot 0\Delta \), where \( f(\alpha) = \frac{\alpha \Delta_0}{0 \Delta_0} \).

- For example, for a triangular membership function, we have \( f(\alpha) = 1 - \alpha \).

- So, if we know the type \( \mu_0 \) (hence \( f(\alpha) \)), and we know the 0-cut, we can compute all \( \alpha \)-cuts as \( \alpha \Delta = f(\alpha) \cdot 0\Delta \).

- So, if \( \mu_i(\Delta x_i) \) and \( \mu_m(\Delta m) \) are of the same type, then \( \alpha \Delta_i = f(\alpha) \cdot 0\Delta_i \) and \( \alpha \Delta_m = f(\alpha) \cdot 0\Delta_m \) for all \( \alpha \).
7. When Simplification Is Possible (cont-d)

- We know that $\alpha \Delta = \sum_{i=1}^{n} |c_i| \cdot \alpha \Delta_i + \alpha \Delta_m$.

- For $\alpha \Delta_i = f(\alpha) \cdot 0 \Delta_i$ and $\alpha \Delta_m = f(\alpha) \cdot 0 \Delta_m$, we get

$$\alpha \Delta = \sum_{i=1}^{n} |c_i| \cdot f(\alpha) \cdot 0 \Delta_i + f(\alpha) \cdot 0 \Delta_m.$$  

- So, $\alpha \Delta = f(\alpha) \cdot \left( \sum_{i=1}^{n} |c_i| \cdot 0 \Delta_i + 0 \Delta_m \right) = f(\alpha) \cdot 0 \Delta$.

- Thus, if all $\mu(x)$ are of the same type $\mu_0(z)$, there is no need to compute $\alpha \Delta$ eleven times:

  - it is sufficient to compute $0 \Delta$;
  
  - to find all other values $\alpha \Delta$, we simply multiply $0 \Delta$ by the factors $f(\alpha)$ corresponding to $\mu_0(z)$.  

8. A More General Case

• A more general case is:
  – when we have a list of $T$ different types of uncertainty – i.e., types of membership functions, and
  – each approximation error $\Delta x_i$ consists of $\leq T$ components of the corresponding type $t$:
    \[
    \Delta x_i = \sum_{t=1}^{T} \Delta x_{i,t} \quad \text{and} \quad \Delta m = \sum_{t=1}^{T} \Delta m_t.
    \]

• For example:
  – type $t = 1$ may correspond to intervals (which are, of course, a particular case of fuzzy uncertainty),
  – type $t = 2$ may correspond to triangular membership functions, etc.
9. How This Case Is Processed Now

• **First stage:**
  
  – we use the known membership functions \( \mu_{i,t}(\Delta x_{i,t}) \)
  and \( \mu_{m,t}(\Delta m_t) \)
  
  – to find the memberships functions \( \mu_i(\Delta x_i) \) and
  \( \mu_m(\Delta m) \) that correspond to the sums \( \Delta x_i \) and \( \Delta m \).

• **Second stage:** we use \( \mu_i(\Delta x_i) \) and \( \mu_m(\Delta m) \) to compute
  the desired membership function \( \mu(\Delta y) \).

• **Problem:** on the second stage, we apply the above formula eleven times:

\[
\alpha \Delta = \sum_{i=1}^{n} |c_i| \cdot \alpha \Delta_i + \alpha \Delta_m.
\]
10. Our Main Idea

- We have $\Delta y = \sum_{i=1}^{n} c_i \cdot \Delta x_i - \Delta m$, where

$$\Delta x_i = \sum_{t=1}^{T} \Delta x_{i,t} \text{ and } \Delta m = \sum_{t=1}^{T} \Delta m_t.$$ 

- Thus, $\Delta y = \sum_{i=1}^{n} c_i \cdot \left( \sum_{t=1}^{T} \Delta x_{i,t} \right) - \left( \sum_{t=1}^{T} \Delta m_t \right).$

- Grouping together all the terms corr. to type $t$, we get $\Delta y = \sum_{t=1}^{T} \Delta y_t$, where $\Delta y_t \overset{\text{def}}{=} \sum_{i=1}^{n} c_i \cdot \Delta x_{i,t} - \Delta m_t$.

- For each $t$, we are combining membership functions of the same type, so it is enough to compute $0 \Delta_t$.

- Then, we add the resulting membership functions – by adding the corresponding $\alpha$-cuts.
11. Resulting Algorithm

- Let \([[-0^\Delta i,t, 0^\Delta i,t]]\) and \([-0^\Delta m,t, 0^\Delta m,t]]\) be 0-cuts of the membership functions \(\mu_{i,t}(\Delta x_{i,t})\) and \(\mu_{m,t}(\Delta m_t)\).

- Based on these 0-cuts, we compute, for each type \(t\), the values \(0^\Delta = \sum_{i=1}^{n} |c_i| \cdot 0^\Delta i,t + 0^\Delta m,t\).

- Then, for \(\alpha = 0, 0.1, \ldots\), and for \(\alpha = 1.0\), we compute the values \(\alpha^\Delta t = f_t(\alpha) \cdot 0^\Delta t\).

- Finally, we add up \(\alpha\)-cuts corresponding to different types \(t\), to come up with the expression \(\alpha^\Delta = \sum_{t=1}^{T} \alpha^\Delta t\).

- Comment. We can combine the last two steps into a single step: \(\alpha^\Delta = \sum_{t=1}^{T} f_t(\alpha) \cdot 0^\Delta t\).
12. The New Algorithm Is Much Faster

- The original algorithm computed the above formula eleven times:

\[
\alpha \Delta = \sum_{i=1}^{n} |c_i| \cdot \alpha \Delta_i + \alpha \Delta_m.
\]

- The new algorithm uses the corresponding formula \( T \) times, i.e., as many times as there are types.

- All the other computations are much faster, since they do not grow with the input size \( n \).

- Thus, if the number \( T \) of different types is smaller than eleven, the new methods is much faster.

- Example: for \( T = 2 \) types (e.g., intervals and triangular \( \mu(x) \)), we get a \( \frac{11}{2} = 5.5 \) times speedup.
13. Conclusions and Future Work

- We can therefore conclude that sometimes, it is beneficial to process different types of uncertainty separately.
- Namely, it is beneficial when we have ten or fewer different types of uncertainty.
- The fewer types of uncertainty we have, the faster the resulting algorithm.
- We plan to test this idea of several actual data processing examples.
- We also plan to extend this idea to other types of uncertainty, in particular, to probabilistic uncertainty.
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