How Resilient Modulus of a Pavement Depends on Moisture Level: Towards a Theoretical Justification of a Practically Important Empirical Formula

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1. What Is a Resilient Modulus: A Brief Reminder

- A road pavement must be sufficiently stiff:
  - when heavy trucks pass over the pavement at a high speed,
  - the pavement’s deformation must be within certain (small) bounds.

- A usual characteristic of stiffness is *modulus*, the ratio of stress (force per area) to deformation.

- For a fixed force, the larger the modulus, the smaller the deformation.

- Force applied to the pavement mostly comes from the fast moving vehicles.

- The effect of such a force can be characterized by a special type of modulus called *resilient modulus*. 
2. Dependence on Moisture Is Important

- To increase stiffness, the pavement is compacted.
- After the compaction, it is necessary to check
  - whether the resilient modulus has reached the desired value,
  - or we need to perform additional compaction (or add an additional pavement layer).
- Ideally, we should make this decision right away, when the equipment is still in the road segment.
- The problem is that after the compaction, it often takes several days for moisture to reach equilibrium.
3. Dependence on Moisture Is Important (cont-d)

- During these days, the equipment has been moved to new road segments; so:
  - if we perform the measurements only after this equilibrium is attained,
  - and it turns out that further work is needed,
  - we will then need to move the equipment back,
  - which requires extra time and extra expenses.
- It is therefore desirable:
  - given the resilient modulus under the current moisture level,
  - to predict the future equilibrium value of the resilient modulus.
4. An Empirical Formula

- Moisture level is characterized by the ratio
  \[ S = \frac{\text{volume of water}}{\text{overall volume of the empty space}}. \]
  - no moisture at all corresponds to \( S = 0.0 \), while
  - \( S = 1.0 \) means that all the empty spaces are filled with water.

- \( S \) is called *degree of saturation*.

- The following formula describes the dependence of the resilient modulus \( M(S) \) on the moisture level \( S \):
  \[
  \ln \left( \frac{M(S)}{M(S_0)} \right) = a + \frac{b - 1}{1 + \exp(\beta - k \cdot (S - S_0))}.
  \]

- *Problem*: this formula is purely empirical and is, thus, not fully reliable.

- *We need*: a theoretical explanation.
5. Scale Invariance

- The numerical value of a physical quantity depends on the choice of a measuring unit.
- If we use a different measuring unit, then all the numerical values change.
- Specifically:
  - if we replace the original measuring unit with a new unit which is $\lambda$ times smaller,
  - then all numerical values get *re-scaled*: namely, they are multiplied by $\lambda$: $x \rightarrow x' = \lambda \cdot x$.
- For example, 3 m becomes $3 \cdot 100 = 300$ cm.
6. Scale Invariance (cont-d)

- In most physical situations, the choice of a measuring unit is rather irrelevant.
- Thus, the corresponding formulas should not change if we use different units.
- Of course, if we describe a dependence $y = f(x)$ and
  - we change a unit for measuring $x$,
  - then we may have to according also change the measuring unit for $y$.
- In the new units for $x$ and $y$, the dependence should look exactly the same as in the old units.
7. Shift Invariance

- For some quantities, the starting point can be arbitrarily chosen.
- We can start measuring time at any moment – we can start at year 0, or we can take year 2016 as year 0.
- If we replace the original starting point with a new one which is $x_0$ unit earlier, then:
  - for the same moment of time,
  - the original numerical value $x$ gets replaced by a new shifted value $x' = x + x_0$.
- It is reasonable to require that if we change the starting point, the dependencies should remain the same.
- We will use these two invariances to derive the desired empirical formula.
8. Using Scale Invariance

- Our general problem is to find how the resilient modulus $M(S)$ depend on the moisture level $S$.
- Let us fix two values $S_1$ and $S_2$, and analyze how to predict $M(S_2)$ based on $M(S_1)$: $M(S_2) = f(M(S_1))$.
- If we re-scale the resilient modulus, we get
  $$M'(S_1) = \lambda \cdot M(S_1) \text{ and } M'(S_2) = \lambda \cdot M(S_2).$$
- We want to check that $M(S_2) = f(M(S_1))$ implies that $M'(S_2) = f(M'(S_1))$, i.e., $\lambda \cdot M(S_2) = f(\lambda \cdot M(S_1))$.
- Substituting $M(S_2) = f(M(S_1))$ into this formula, we conclude that $f(\lambda \cdot M(S_1)) = \lambda \cdot f(M(S_1))$.
- For any $x$, we can take $\lambda = x$ and $M(S_1) = 1$, and get $f(x) = x \cdot f(1)$, i.e., $f(x) = c \cdot x$ for some constant $c$.
- Thus, $M(S_2) = c(S_1, S_2) \cdot M(S_1)$. Let us find the dependence $c(S_1, S_2)$. 
9. **Shift Invariance**

- Moisture level is a unit-less quantity, there is no easy way to change a measuring unit.
- However, the starting point for moisture level can be chosen differently.
- We can start with the absolute zero moisture level.
- This level is difficult to obtain in practice.
- Alternatively, we can use a realistic level $S_0$ as a starting point.
- It is reasonable to require that the formulas should not change if we replace $S$ by $S' = S + S_0$. 
10. Using Shift Invariance: First Try

- A natural idea is to require that \( c(S_1, S_2) \) does not change after shift: \( c(S_1 + S_0, S_2 + S_0) = c(S_1, S_2) \).

- In particular, for \( S_0 = -S_2 \), we get \( c(S_1, C_2) = C(S_1 - S_2) \), where \( C(S) \overset{\text{def}}{=} c(S, 0) \).

- For every \( S_1 \) and \( S_2 \), we have \( M(S_1) = C(S_1) \cdot M(0) \) and \( M(S_1 + S_2) = C(S_2) \cdot M(S_1) \) and thus,
  \[
  M(S_1 + S_2) = C(S_2) \cdot C(S_1) \cdot M(0).
  \]

- On other other hand, we can directly apply the shift by \( S_1 + S_2 \), then we get
  \[
  M(S_1 + S_2) = C(S_1 + S_2) \cdot M(0).
  \]

- By comparing the two formulas for \( M(S_1 + S_2) \), we conclude that \( C(S_1 + S_2) = C(S_2) \cdot C(S_1) \).
11. First Try (cont-d)

• Reminder:

\[ C(S_1 + S_2) = C(S_2) \cdot C(S_1). \]

• In physics, most dependencies are continuous.

• Each continuous functions satisfying the above functional equation have the form \( C(S) = \exp(k \cdot S) \), so:

\[ M(S) = c(S, 0) \cdot M(0) = C(S) \cdot M(0) = \exp(k \cdot S) \cdot M(0). \]

• Alas, this formula does not fit well with empirical data.

• Thus, we need to use invariances in a more subtle way.
12. Second Idea

- **Idea:**
  - if we know $M(S)$
  - and we know some moisture-independent value $M$ (e.g., the largest possible value of resilient modulus corresponding to all possible moisture levels),
  - then we should be able to reconstruct $M(S + S_0)$.

- In other words, we have $M(S + S_0) = f(M(S), M)$ for some function $f(x, y)$ depending only on $S_0$.

- The dependence $f(x, y)$ should be scale-invariant.

- Let us take into account that we may have different procedures for measuring $M$ and $M(S)$.

- Thus, in principle, we may have two different measuring units used in these measurements.
13. Second Idea (cont-d)

• It is therefore reasonable to require that:
  • we can change the units of both quantities
  • and still get the same dependence,
  • after, possibly, appropriately re-scaling the resulting value \( M(S + S_0) \).

• In other words, for every two factors \( \lambda \) and \( \mu \), we have a factor \( \nu(\lambda, \mu) \) for which \( M(S + S_0) = f(M(S), M) \) implies

\[
f(\lambda \cdot M(S), \mu \cdot M) = \nu(\lambda, \mu) \cdot f(M(S), M).
\]

• It is known that every continuous function \( f(x, y) \) with this property has the form \( f(x, y) = A \cdot x^\alpha \cdot y^\beta \).

• Thus, \( M(S + S_0) = A \cdot (M(S))^\alpha \cdot M^\beta \).
14. Second Idea (final)

- For $M(S) = C(S) \cdot M$ and $M(S + S_0) = C(S + S_0) \cdot M$, we get $C(S + S_0) \cdot M = A \cdot (C(S))^\alpha \cdot M^\alpha \cdot M^\beta$.

- For $M = 1$, we get $C(S + S_0) = A \cdot (C(S))^\alpha$, where $A$ and $\alpha$ depend on $S_0$.

- By taking logarithms of both sides, we conclude that for the function $L(S) \overset{\text{def}}{=} \ln(C(S))$, we get

  $$L(S+S_0) = a(S_0) + \alpha(S_0) \cdot L(S), \text{ where } a(S_0) \overset{\text{def}}{=} \ln(A(S_0)).$$

- So, the value $L(S + S_0)$ is obtained from $L(S)$ by a linear transformation $L(S) \rightarrow a(S_0) \cdot \alpha(S_0) \cdot L(S)$.

- Functions satisfying the above functional equation are also known known.

- Alas, these functions are also not a perfect fit for the empirical data.
15. Final Idea

- We know that linear transformations are not sufficient.
- Let us thus consider the possibility that the transformation from $L(S)$ to $L(S + S_0)$ may be non-linear.
- To be precise, we are looking for a group of transformations – i.e., set of transformations which is:
  - closed under composition and
  - closed under taking the inverse.
- This group must:
  - include all linear transformations and
  - be not too large – e.g., be finite-dimensional.
- This means that each transformation can be uniquely determined by values of finitely many parameters.
- It turns out that, under certain reasonable continuity requirements, all such transformations are known.
16. Final Idea (cont-d)

- Specifically, all such transformations are fractional-linear:

\[ L(S) \rightarrow \frac{a \cdot L(S') + b}{c \cdot L(S) + d}. \]

- We are thus looking for functions \( L(S') \) for which

\[ L(S + S_0) = \frac{a(S_0) \cdot L(S') + b(S_0)}{c(S_0) \cdot L(S) + d(S_0)}. \]

- Such functions are also known:
  - under reasonable monotonicity conditions,
  - they get exactly the desired form.

- Thus, we indeed get the desired justification for the above empirical formula.
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