Kinematic Spaces: Motivations from Space-Time Physics, Main Mathematical Results, Algorithmic Results and Challenges, and Possible Relation to de Vries Algebras

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1. Why Ordering Relations

• Traditionally, in physics, space-times are described by (pseudo-)Riemann spaces, i.e.:
  – by smooth manifolds
  – with a tensor metric field $g_{ij}(x)$.

• However, in several physically interesting situations smoothness is violated and metric is undefined:
  – near the singularity (Big Bang),
  – at the black holes, and
  – on the microlevel, when we take into account quantum effects.

• In all these situations, what remains is causality $\preceq$ – an ordering relation.

2. Causality: Brief History

- In Newton’s physics, signals can potentially travel with an arbitrarily large speed.

- Let $a = (t, x)$ denote an event occurring at the spatial location $x$ at time $t$.

- Then, an event $a = (t, x)$ can influence an event $a' = (t', x')$ if and only if $t \leq t'$.

- The fundamental role of the non-trivial causality relation emerged with the Special Relativity (SRT).

- In SRT, the speed of all the signals is limited by the speed of light $c$.

- As a result, $a = (t, x) \leq a' = (t', x')$ if and only if $t' \geq t$ and $\frac{d(x, x')}{t' - t} \leq c$, i.e.:

$$c \cdot (t' - t) \geq \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}.$$
3. Causality: A Graphical Description

\[ x = -c \cdot t \]

\[ x = c \cdot t \]
4. Importance of Causality

- In the original special relativity theory, causality was just one of the concepts.

- Its central role was revealed by A. D. Alexandrov (1950) who showed that in SRT, causality implied Lorenz group:

- Every order-preserving transforming of the corr. partial ordered set is linear, and is a composition of:
  - spatial rotations,
  - Lorentz transformations (describing a transition to a moving reference frame), and
  - re-scalings $x \rightarrow \lambda \cdot x$ (corresponding to a change of unit for measuring space and time).

- This theorem was later generalized by E. Zeeman and is known as the Alexandrov-Zeeman theorem.
5. When is Causality Experimentally Confirmable?

- In many applications, we only observe an event \( b \) with some accuracy.

- For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction.

- However, in practice, we cannot measure time exactly, so, we observe an event occurring \( 1 \pm 0.001 \) sec after \( a \).

- In general, we can only guarantee that the observed event is within a certain neighborhood \( U_b \) of the event \( b \).

- Because of this uncertainty, the only possibility to experimentally confirm that \( a \) can influence \( b \) is when

  \[
  a \prec b \iff \exists U_b \forall \tilde{b} \in U_b \ (a \preceq \tilde{b}).
  \]

- In topological terms, this means that \( b \) is in the interior \( K_a^+ \) of the closed cone \( C_a^+ = \{c : a \preceq c\} \).
6. Kinematic Orders

- In physics, $a \prec b$ correspond to influences with speeds smaller than the speed of light.

- There are two types of objects:
  - objects with non-zero rest mass can travel with any possible speed $v < c$ but not with the speed $c$;
  - objects with zero rest mass (e.g., photons) can travel only with the speed $c$, but not with $v < c$.

- Thus, $\prec$ correspond to causality by traditional (kinematic) objects.

- Because of this:
  - the relation $\prec$ is called *kinematic causality*, and
  - spaces with this relation $\prec$ are called *kinematic spaces*. 
7. Kinematic Spaces: Towards a Description

- To describe space-time, we thus need a (pre-)ordering relation ≼ (causality) and topology (equality closeness).
- Natural continuity: for every event \( a \) and for every neighborhood \( U_a \), there exist \( a^- \prec a \) and \( a \prec a^+ \).
- Natural topology: every neighborhood \( U_a \) contains an open interval \((a', a'') = \{b : a' \prec c \prec a''\}\).
- Natural idea: a motion with speed \( c \) is a limit of motions with speeds \( v < c \) when \( v \to c \).
- Resulting description of \( \preceq \) in terms of \( \prec \): \( C^+ = \overline{K}^+ \) and \( C^- = \overline{K}^- \), i.e., \( b \succ a \iff \forall U_b \exists \tilde{b} \left( \tilde{b} \in U_b \& \tilde{b} \succ a \right) \).
- For \( U_b = (b', b'') \), when \( b \prec b'' \), we get \( a \prec \tilde{b} \prec b'' \) hence \( a \prec b'' \).
- Thus, \( a \preceq b \iff \forall c(b \prec c \Rightarrow a \prec c) \).
8. Resulting Definition

- A set $X$ with a partial order $\prec$ is called a kinematic space if it satisfies the following conditions:

  $\forall a \exists a_-, a_+ (a_- \prec a \prec a_+)$;
  $\forall a, b (a \prec b \rightarrow \exists c (a \prec c \prec b))$;
  $\forall a, b, c (a \prec b, c \rightarrow \exists d (a \prec d \prec b, c))$;
  $\forall a, b, c (b, c \prec a \rightarrow \exists d (b, c \prec d \prec a))$.

- We take a topology generated by intervals $(a, b) = \{c : a \prec c \prec b\}$.

- A kinematic space is called normal if

  $b \in \{c : c \succ a\} \iff a \in \{c : c < b\}$.

- For a normal kinematic space, we denote $b \in \{c : c \succ a\}$ by $a \preceq b$.

- It is proven that $a \prec b \preceq c$ and $a \preceq b \prec c$ imply $a \prec c$. 

9. First Result: Reconstructing $\prec$ from $\preceq$

- We consider separable kinematic spaces.
- We say that a space is *complete* if every $\preceq$-decreasing bounded sequence $\{s_n\}$ has a limit, i.e., $\land s_n$.

- *Lemma.* If every closed intervals $\{c : a \preceq c \preceq b\}$ is compact, then the space is complete.

- If two complete separable normal kinematic orders $\prec$ and $\prec'$ on $X$ lead to the same closed order $\preceq = = \preceq'$, then $\prec = \prec'$.

- Let $S_e$ denote the set of all $\preceq$-decreasing sequences $s = \{s_n\}$ for which $\land s_n = e$.

- For $s, s' \in S_e$, we define $s \succeq s' \iff \forall n \exists m (s_n \succeq s'_m)$; then:

$$a \succ b \iff \exists e \succeq b \exists s = \{s_n\} \in S_e (s \text{ is largest in } S_e \& s_1 = a).$$
10. Symmetry: a Fundamental Property of the Physical World

• One of the main objectives of science: prediction.

• *Basis for prediction*: we observed similar situations in the past, and we expect similar outcomes.

• *In mathematical terms*: similarity corresponds to symmetry, and similarity of outcomes – to invariance.

• *Example*: we dropped the ball, it fall down.

• *Symmetries*: shift, rotation, etc.

• *In modern physics*: theories are usually formulated in terms of symmetries (not diff. equations).

• *Natural idea*: let us use symmetry to analyze causality as well.
11. Static Space-Times: Space-Times Invariant under Arbitrary Temporal Shifts

Let \( \{T_t\}, t \in \mathbb{R} \) be a 1-parametric group of \( \preceq \)-preserving transformations on a (pre-)ordered set \((E, \preceq)\).

We require that:

- if \( t > 0 \), then \( \forall e \ (e \preceq T_t(e) \& T_t(e) \not\preceq e) \);
- if \( t_n \to t \) and \( \forall n \ (e \preceq T_{t_n}(e')) \), then \( e \preceq T_t(e') \);
- for every \( e, e' \in E \), there exists a \( t \) for which \( e \preceq T_t(e') \) (no cosmological (particle) horizons).

Then, there exists a set \( X \) with a function \( d : X \times X \to \mathbb{R} \) for which \( E \approx \mathbb{R} \times X \) with

\[(t, x) \preceq (t', x') \iff t' - t \geq d(x, x').\]

We can have a metric space \((X, d) \iff \text{the space } E \text{ is } T\text{-invariant w.r.t. some } T : E \to E \text{ s.t. } T^2 = \text{id and } a \preceq b \iff T(b) \preceq T(a).\)
12. de Vries Algebras: Definition

A de Vries algebra is a pair consisting of a complete Boolean algebra \((B, \preceq)\) and a binary relation \(<\) (proximity) for which:

- \(1 < 1\);
- \(a < b\) implies \(a \preceq b\);
- \(a \preceq b < c \preceq d\) implies \(a < d\);
- \(a < b, c\) implies \(a < b \land c\);
- \(a < b\) implies \(\neg b < \neg a\);
- \(a < b\) implies there exists \(c\) such that \(a < c < b\);
- \(a \neq 0\) implies there exists \(b \neq 0\) such that \(b < a\).
13. Relation Between Kinematics and de Vries Algebras

- We say that a de Vries algebra is connected if \( a \prec a \) implies that \( a = 0 \) or \( a = 1 \).

- For every connected de Vries algebra \( B \):
  - the set \( B - \{0, 1\} \) with a proximity relation \( \prec \) is a normal kinematic space, and
  - the original relation \( \preceq \) coincides with the closure of \( \prec \) in the sense of kinematic spaces.

- Let \( S \) be a normal kinematic space with anti-tonic mapping \( \neg \) for which \( \neg
\neg a = a \) and for which,
  - if we add 0 and 1 to the corresponding set \( (S, \preceq) \),
  - we get a complete Boolean algebra.

Then, this set is also a connected de Vries algebra.
14. Product Operations for Posets: Examples

- Let us assume that we have two parallel (independent) universes $A_1$ and $A_2$.
- Then an event in a multi-verse is a pair $(a_1, a_2)$, where $a_1 \in A_1$ and $a_2 \in A_2$.
- To compare such pairs, we must therefore define a partial order on the set $A_1 \times A_2$ of all such pairs.
- For independent universes, a natural definition is a Cartesian product:

  $$(a_1, a_2) \leq (a'_1, a'_2) \iff ((a_1 \leq a'_1) \& (a_2 \leq a'_2)).$$

- Another operation is a lexicographic product:

  $$(a_1, a_2) \leq (a'_1, a'_2) \iff$$

  $$((a_1 \leq a'_1) \& a_1 \neq a'_1) \lor ((a_1 = a'_1) \& (a_2 \leq a'_2))).$$
15. Possible Physical Meaning of Lexicographic Order

**Idea:**

- $A_1$ is *macroscopic* space-time,
- $A_2$ is *microscopic* space-time:

\[
(a_1, a_2) \quad a_1 \\
(a_1, a'_2) \quad a_1
\]

\[
(a'_1, a_2) \quad a'_1
\]
16. Natural Questions

- **Question**: when does the resulting partially ordered set $A_1 \times A_2$ satisfy a certain property?

- **Examples**: is it a total order? is it a lattice order?

- **It is desirable** to reduce the question about $A_1 \times A_2$ to questions about properties of component spaces $A_i$.

- **Some such reductions are known**; e.g.:
  - A Cartesian product is a total order $\iff$ one of $A_i$ is a total order, and the other has only one element.
  - A lexicographic product is a total order if and only if both components are totally ordered.

- **In this talk**, we provide a general algorithm for such reduction.
17. Similar Questions in Other Areas

- Similar questions arise in other applications of ordered sets.
- *Our algorithm* does not use the fact that the original relations are orders.
- Thus, our algorithm is applicable to a *general* binary relation – equivalence, similarity, etc.
- Moreover, this algorithm can be applied to the case when we have a space with *several* binary relations.
- *Example:* we may have an order relation and a similarity relation.
18. Definitions

- **By a space**, we mean a set $A$ with $m$ binary relations $P_1(a, a')$, \ldots, $P_m(a, a')$.

- **By a 1st order property**, we mean a formula $F$ obtained from $P_i(x, x')$ by using logical $\lor$, $\land$, $\neg$, $\rightarrow$, $\exists x$ and $\forall x$.

- **Note**: most properties of interest are 1st order; e.g. to be a total order means $\forall a \forall a' ((a \leq a') \lor (a' \leq a))$.

- **By a product operation**, we mean a collection of $m$ propositional formulas that
  - describe the relation $P_i((a_1, a_2), (a'_1, a'_2))$ between the elements $(a_1, a_2), (a'_1, a'_2) \in A_1 \times A_2$
  - in terms of the relations between the components $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$ of these elements.

- **Note**: both Cartesian and lexicographic order are product operations in this sense.
19. Main Result

• **Main Result.** *There exists an algorithm that, given*
  
  • *a product operation and*
  
  • *a property* $F$,

  *generates a list of properties* $F_{11}, F_{12}, \ldots, F_{p1}, F_{p2}$ *s.t.*:

  \[
  F(A_1 \times A_2) \iff ((F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor (F_{p1}(A_1) \& F_{p2}(A_2))).
  \]

• **Example:** For Cartesian product and total order $F$, we have

  \[
  F(A_1 \times A_2) \iff ((F_{11}(A_1) \& F_{12}(A_2)) \lor (F_{21}(A_1) \& F_{22}(A_2))) :
  \]

  • $F_{11}(A_1)$ means that $A_1$ is a total order,

  • $F_{12}(A_2)$ means that $A_2$ is a one-element set,

  • $F_{21}(A_1)$ means that $A_1$ is a one-element set, and

  • $F_{22}(A_2)$ means that $A_2$ is a total order.
20. Auxiliary Results

- **Generalization:**
  - A similar algorithm can be formulated for a product of three or more spaces.
  - A similar algorithm can be formulated for the case when we allow ternary and higher order operations.

- **Specifically for partial orders:**
  - The only product operations that always leads to a partial order on \( A_1 \times A_2 \) for which
    \[
    (a_1 \leq_1 a_1' \& a_2 \leq_2 a_2') \rightarrow (a_1, a_2) \leq (a_1', a_2')
    \]
    are Cartesian and lexicographic products.
21. Proof of the Main Result

- The desired property $F(A_1 \times A_2)$ uses:
  - relations $P_i(a, a')$ between elements $a, a' \in A_1 \times A_2$;
  - quantifiers $\forall a$ and $\exists a$ over elements $a \in A_1 \times A_2$.

- Every element $a \in A_1 \times A_2$ is, by definition, a pair $(a_1, a_2)$ in which $a_1 \in A_1$ and $a_2 \in A_2$.

- Let us explicitly replace each variable with such a pair.

- By definition of a product operation:
  - each relation $P_i((a_1, a_2), (a'_1, a'_2))$
  - is a propositional combination of relations betw. elements $a_1, a'_1 \in A_1$ and betw. elements $a_2, a'_2 \in A_2$.

- Let us perform the corresponding replacement.

- Each quantifier can be replaced by quantifiers corresponding to components: e.g., $\forall(a_1, a_2) \iff \forall a_1 \forall a_2$. 
22. Proof of the Main Result (cont-d)

- So, we get an equivalent reformulation of $F$ s.t.:
  - elementary formulas are relations between elements of $A_1$ or between $A_2$, and
  - quantifiers are over $A_1$ or over $A_2$.
- We use induction to reduce to the desired form
  \[(F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor (F_{p1}(A_1) \& F_{p2}(A_2))\).
- Elementary formulas are already of the desired form – provided, of course, that we allow free variables.
- We will show that:
  - if we apply a propositional connective or a quantifier to a formula of this type,
  - then we can reduce the result again to the formula of this type.
23. Applying Propositional Connectives

- We apply propositional connectives to formulas of the type
  \[(F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor (F_{p1}(A_1) \& F_{p2}(A_2))\].
- We thus get a propositional combination of the formulas of the type \(F_{ij}(A_j)\).
- An arbitrary propositional combination can be described as a disjunction of conjunctions (DNF form).
- Each conjunction combines properties related to \(A_1\) and properties related to \(A_2\), i.e., has the form
  \[G_1(A_1) \& \ldots \& G_p(A_1) \& G_{p+1}(A_2) \& \ldots \& G_q(A_2)\].
- Thus, each conjunction has the from \(G(A_1) \& G'(A_2)\), where \(G(A_1) \Leftrightarrow (G_1(A_1) \& \ldots \& G_p(A_1))\).
- Thus, the disjunction of such properties has the desired form.
24. Applying Existential Quantifiers

- When we apply $\exists a_1$, we get a formula
  $$\exists a_1 ((F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor (F_{p1}(A_1) \& F_{p2}(A_2))).$$

- It is known that $\exists a (A \lor B)$ is equivalent to $\exists a A \lor \exists a B$.

- Thus, the above formula is equivalent to a disjunction
  $$\exists a_1 (F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor \exists a_1 (F_{p1}(A_1) \& F_{p2}(A_2)).$$

- Thus, it is sufficient to prove that each formula
  $$\exists a_1 (F_{i1}(A_1) \& F_{i2}(A_2))$$
  has the desired form.

- The term $F_{i2}(A_2)$ does not depend on $a_1$ at all, it is all about elements of $A_2$.

- Thus, the above formula is equivalent to
  $$(\exists a_1 F_{i1}(A_1)) \& F_{i2}(A_2).$$

- So, it is equivalent to the formula $F'_{i1}(A_1) \& F_{i2}(A_2)$, where $F'_{i1} \Leftrightarrow \exists a_1 F_{i1}(A_1)$. 
25. Applying Universal Quantifiers

- When we apply a universal quantifier, e.g., $\forall a_1$, then we can use the fact that $\forall a_1 F$ is equivalent to $\neg \exists a_1 \neg F$.

- We assumed that the formula $F$ is of the desired type

$$ (F_{11}(A_1) \& F_{12}(A_2)) \lor \ldots \lor (F_{p1}(A_1) \& F_{p2}(A_2)). $$

- By using the propositional part of this proof, we conclude that $\neg F$ can be reduced to the desired type.

- Now, by applying the $\exists$ part of this proof, we conclude that $\exists a_1 (\neg F)$ can also be reduced to the desired type.

- By using the propositional part again, we conclude that $\neg (\exists a_1 \neg F)$ can be reduced to the desired type.

- By induction, we can now conclude that the original formula can be reduced to the desired type.

- The main result is proven.
26. Example of Applying the Algorithm

- Let us apply our algorithm to checking whether a Cartesian product is totally ordered.
- In this case, $F$ has the form $\forall a \forall a' ((a \leq a') \lor (a' \leq a))$.
- We first replace each variable $a, a' \in A_1 \times A_2$ with the corresponding pair:

$$\forall (a_1, a_2) \forall (a'_1, a'_2) (((a_1, a_2) \leq (a'_1, a'_2)) \lor ((a'_1, a'_2) \leq (a_1, a_2))).$$

- Replacing the ordering relation on the Cartesian product with its definition, we get

$$\forall (a_1, a_2) \forall (a'_1, a'_2) ((a_1 \leq a'_1 \land a_2 \leq a'_2) \lor (a'_1 \leq a_1 \land a'_2 \leq a_2)).$$

- Replacing $\forall a$ over pairs with individual $\forall a_i$, we get:

$$\forall a_1 \forall a_2 \forall a'_1 \forall a'_2 (((a_1 \leq a'_1 \land a_2 \leq a'_2)) \lor ((a'_1 \leq a_1 \land a'_2 \leq a_2))).$$

- By using the $\forall \iff \neg \exists \neg$, we get an equivalent form

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \neg(((a_1 \leq a'_1 \land a_2 \leq a'_2) \lor (a'_1 \leq a_1 \land a'_2 \leq a_2))).$$
27. Example (cont-d)

- So far, we got:
  \[ \neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \neg ((a_1 \leq a'_1 \land a_2 \leq a'_2) \lor (a'_1 \leq a_1 \land a'_2 \leq a_2)) \].

- Moving \( \neg \) inside the propositional formula, we get
  \[ \neg \exists a_1 \exists a_1' \exists a_2' \exists a'_2' \neg ((a_1 \not\leq a'_1 \lor a_2 \not\leq a'_2) \land (a'_1 \not\leq a_1 \lor a'_2 \not\leq a_2)) \].

- The formula \((a_1 \not\leq a'_1 \lor a_2 \not\leq a'_2) \land (a'_1 \not\leq a_1 \lor a'_2 \not\leq a_2)\) must now be transformed into a DNF form.

- The result is \((a_1 \not\leq a'_1 \land a_2 \not\leq a'_2) \lor (a'_1 \not\leq a_1 \land a'_2 \not\leq a_2) \lor (a_2 \not\leq a'_2 \land a'_1 \not\leq a_1) \lor (a_2 \not\leq a'_2 \land a'_2 \not\leq a_2)\).

- Thus, our formula is \(\iff \neg (F_1 \lor F_2 \lor F_3 \lor F_4)\), where
  \[ F_1 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 (a_1 \not\leq a'_1 \land a'_1 \not\leq a_1), \]
  \[ F_2 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 (a_1 \not\leq a'_1 \land a'_2 \not\leq a_2), \]
  \[ F_3 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 (a_2 \not\leq a'_2 \land a'_1 \not\leq a_1), \]
  \[ F_4 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 (a_2 \not\leq a'_2 \land a'_2 \not\leq a_2). \]
28. Example (cont-d)

- So far, we got \( \iff \neg (F_1 \lor F_2 \lor F_3 \lor F_4) \), where
  
  \[
  F_1 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \ (a_1 \nleq a'_1 \& a'_1 \nleq a_1),
  \]
  
  \[
  F_2 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \ (a_1 \nleq a'_1 \& a'_2 \nleq a_2),
  \]
  
  \[
  F_3 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \ (a_2 \nleq a'_2 \& a'_1 \nleq a_1),
  \]
  
  \[
  F_4 \iff \exists a_1 \exists a_2 \exists a'_1 \exists a'_2 \ (a_2 \nleq a'_2 \& a'_2 \nleq a_2).
  \]

- By applying the quantifiers to the corresponding parts of the formulas, we get
  
  \[
  F_1 \iff \exists a_1 \exists a'_1 \ (a_1 \nleq a'_1 \& a'_1 \nleq a_1),
  \]
  
  \[
  F_2 \iff (\exists a_1 \exists a'_1 a_1 \nleq a'_1) \& (\exists a_2 \exists a'_2 a'_2 \nleq a_2),
  \]
  
  \[
  F_3 \iff (\exists a_1 \exists a'_1 a'_1 \nleq a_1) \& (\exists a_2 \exists a'_2 a_2 \nleq a'_2),
  \]
  
  \[
  F_4 \iff \exists a_2 \exists a'_1 \exists a'_2 \ (a_2 \nleq a'_2 \& a'_2 \nleq a_2).
  \]

- Then, we again reduce \( \neg (F_1 \lor F_2 \lor F_3 \lor F_4) \) to DNF.

- At present, two product operations are known:
  - *Cartesian product*
    \[(a_1, a_2) \leq (a'_1, a'_2) \iff (a_1 \leq_1 a'_1 \& a_2 \leq_2 a'_2);\]
    and
  - *lexicographic product*
    \[(a_1, a_2) \leq (a'_1, a'_2) \iff ((a_1 \leq_1 a'_1 \& a_1 \neq a'_1) \vee (a_1 = a'_1 \& a_2 \leq_2 a'_2)).\]

- *Question:* what other operations are possible?
30. **Theorem**

- By a *product operation*, we mean a Boolean function
  \[ P : \{T, F\}^4 \rightarrow \{T, F\}. \]

- For every two partially ordered sets \( A_1 \) and \( A_2 \), we define the following relation on \( A_1 \times A_2 \):
  \[
  (a_1, a_2) \leq (a'_1, a'_2) \overset{\text{def}}{=} P(a_1 \leq_1 a'_1, a'_1 \leq_1 a_1, a_2 \leq_2 a'_2, a'_2 \leq_2 a_2).
  \]

- We say that a product operation is *consistent* if \( \leq \) is always a partial order, and
  \[
  (a_1 \leq_1 a'_1 \& a_2 \leq_2 a'_2) \Rightarrow (a_1, a_2) \leq (a'_1, a'_2).
  \]

- **Theorem:** *Every consistent product operation is the Cartesian or the lexicographic product.*
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